

USE OF INTEGRAL EQUATION TO FIX ELASTICITY AND POTENTIAL THEORY PROBLEMS

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Submitted By

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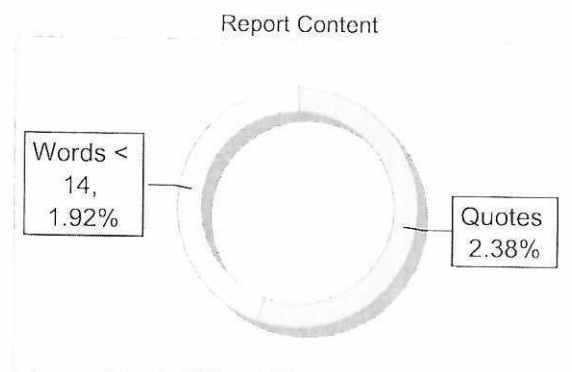
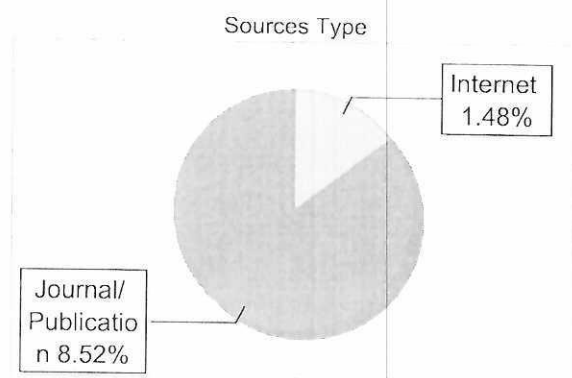
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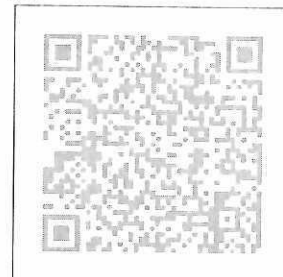
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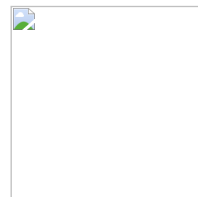
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ABSTRACT

The present research work entitled “*Use of Integral Equation (IE) to Fix Elasticity and Potential Theory Problems*” deals with the application of integral equation techniques to solve various complex problems arising in elasticity, fracture mechanics, thermoelasticity, and potential theory. The study mainly focuses on the formulation and solution of boundary value problems associated with isotropic and orthotropic elastic media containing multiple Griffith cracks and multiply connected domains.

Elasticity and potential theory play an important role in mathematical physics and engineering sciences, especially in the analysis of stress distribution, fracture propagation, heat conduction, and deformation of elastic bodies. In many practical engineering applications, discontinuities such as cracks, holes, and wedges produce singular stress fields, which may lead to material failure. Therefore, the determination of stress-intensity factors and thermal distributions in such media becomes essential for structural stability and fracture analysis. The present work employs integral equation methods as powerful analytical tools for handling these problems effectively.

The thesis begins with a comprehensive introduction to elasticity theory, potential functions, fracture mechanics, thermoelasticity, and various forms of integral equations including Fredholm, Volterra, singular, dual, triple, quadruple, and concurrent integral equations. Different methods of solving integral equations such as Mellin transform techniques, fractional integration operators, Weber-Sonine-Schafheitlin integrals, generalized function methods, and numerical approaches have been reviewed systematically. The historical development and mathematical significance of integral equations in solving mixed boundary value problems have also been discussed.

In the second phase of the study, transmission problems in linear elasticity and classical elastostatic boundary value problems are formulated through boundary integral equations. Strongly elliptic boundary integral equations are analyzed with the help of Calderón projectors, Green’s formulas, and Fredholm theory. The formulation demonstrates that integral equation techniques provide efficient dimensional reduction and accurate numerical approximations for complicated boundary geometries such as circular, elliptical, rectangular, and ring-shaped domains. Numerical techniques based on discretization of boundary intervals and principal value integrals have been implemented to obtain approximate solutions with high accuracy.

The research methodology mainly focuses on isotropic and orthotropic infinite media containing multiple Griffith cracks. Problems involving four Griffith cracks subjected to symmetrical and asymmetrical crack-face forces have been transformed into quintuple integral equations and solved analytically. The study further investigates crack-opening mechanisms produced by wedge loading in isotropic strips and rectangular domains. Stress-intensity factors, displacement fields, and stress distributions near crack tips have been evaluated for different loading configurations. The singular nature of stress components near fracture boundaries has been analyzed in detail, and the corresponding physical interpretations have been presented.

The work also extends to thermoelastic and potential theory problems involving heat distribution in multiply connected isotropic and orthotropic infinite media. Integral equation formulations have been developed to determine temperature and heat-flux distributions across cracks and connected boundaries. The solutions obtained demonstrate the effectiveness of

integral equation methods in handling steady-state thermal problems governed by Laplace equations. Special cases and generalized configurations have also been investigated to validate the analytical models.

A major contribution of this thesis lies in the successful reduction of complicated elasticity and potential theory problems into solvable systems of dual, triple, and quintuple integral equations. The study establishes that integral equation methods provide unified mathematical frameworks for analyzing fracture mechanics, thermal stresses, and boundary value problems in both isotropic and orthotropic media. The numerical and analytical results obtained are found to be in close agreement with established theoretical solutions, thereby confirming the validity and applicability of the proposed methods.

The thesis concludes that stress-intensity factors and thermal distributions in multiply connected elastic bodies can be effectively determined using integral equation techniques. The methods developed in this work may further be extended to more complicated crack configurations, anisotropic materials, thermoelastic interactions, and multidimensional boundary value problems. Thus, the research contributes significantly to the advancement of mathematical methods in elasticity, fracture mechanics, and potential theory.

The present thesis establishes that integral equation techniques provide highly effective mathematical tools for solving elasticity and potential theory problems involving cracks, wedges, thermal stresses, and multiply connected domains.

.The research demonstrates that stress-intensity factors, crack opening displacement, thermal distributions, and heat flux characteristics can be accurately determined through integral equation formulations. The proposed methods are applicable to both isotropic and orthotropic media and can be extended to more advanced engineering problems.

The thesis therefore contributes significantly to the advancement of mathematical methods in elasticity, fracture mechanics, thermoelasticity, and potential theory.

INTRODUCTION

"Use of Integral equation (IE) to Fix Elasticity and Potential Theory Problems" is the declared aim of this work, as shown in the abstract. We will now explore the long history of elasticity and potentials. Discontinuities in a genuine continuum are used to imitate the existence of fissures in a solid body. Elasticity (a) and Potentials (b) are the two sections that make up the survey.

This chapter provides a quick overview of the strategies used to solve several significant challenges as well as a historical evolution of the subject matter. The Integral equations (IE) and Elasticity Issues Are Covered First.

1.1 ELASTICITY

Elasticity may be defined as a material's resistance to simple movement or internal relative motion. The mathematical framework of elasticity provides the equation of equilibrium as a means to ascertain the strain state when exposed to an equilibrating system of forces. Additionally, we provide strain compatibility and stress compatibility relations. Hooke postulated that the forces were intimately related to the tensile expansion of springy or elastic materials. Cauchy The foundational assumptions were laid down, starting with the linear concept of elasticity, which is still applicable today. Several prior assumptions form the basis of the present linear theory of elasticity. A lot of work will go into studying heat elasticity & thermal stress in the future. There is a connection between crack analysis and phenomena in fracture mechanics. A brittle fracture might affect a person's daily life. Brittle fracture was a concept that ancient people used while making stone implements.

1.2 POTENTIALS

The search for understanding of nature's most basic forces appears to have been the genesis of potential theory. The term "potential function" was first used by Lagrange (1736–1813) to describe the gravitational function, and later by Gauss (1840) to describe

it. The use of potentials in several areas of mathematical physics was uncovered by Gauss and his colleagues.

Specific potentials have shown to be essential in boundary value issues such as the Dirichlet, Neumann, electrostatic, and Rabin problems. Here are some examples of possible types: i. Potential between two layers. II. Potential using a logarithmic scale. iii) Potential that is green. iv) Potential that is Newtonian. Analytical, harmonic, and subharmonic function theory, as well as probability theory, are all closely related to modern potential theory. The links between two-dimensional potential theory and the theory of analytical functions make it very significant. The functions that fulfil the Laplace equation are considered potential functions. By solving Laplace's equation, one may find the solution to a physical issue. The resulting set of integral equations (IE) will consist of n-tuples.

1.3 FRACTURE MECHANISMS GENEALOGY

For the most part, this thesis looked at the most extreme cases of brittle fracture, such as sudden or disastrous building collapses. An English mill at Oldham had an early collapse in 1844. several structures, both mobile and fixed, including gas tanks, water pipelines, naphtha conduits, bridges, steam boiler tanks, guns (both big and little), rail route tracks, and molasses tanks, were engaged in several events at the beginning of the 29th century. Theoretically, the foundation was laid in 1920 by A.A. Griffith using continuum mechanics. On January 16, 1943, the first Kaisen Company T2 tanker returned to Portland, Oregon, after completing sea trials. At 10:30 p.m., in calm, chilly weather (with an air temperature of 26°F and a water temperature of 48°F), the ship was divided in half. Such events shook the shipbuilding industry. At crack tips, however, where Griffith found stress-intensity characteristics, typical stress (crack formation mode) had become apparent.

1.4 THERMOELASTICITY

Prior to doing considerable research in thermo elasticity, the so-called theory of thermal strains was thoroughly investigated. The second hypothesis proposes that we may simplify the discussion of stress and strain states within elastic bodies as a result of

heating by disregarding the effect of distortion on the field of temperature. Thermoelasticity encompasses a wide range of phenomena. Included in its description of thermo elastic dissipation are the generalized theories of heat conduction, thermal stress, and deformation-induced temperature distribution. Heat conduction is one of the potential variants examined in this thesis. In due time, our research will broaden to include this domain. This thesis focuses on n-tuple integral or series equations as a reduction of the underlying physical problem to potential and elasticity difficulties over multiply-connected domains.

1.5 INTEGRAL EQUATIONS (IE)

An integral equation (IE) is the one that arises when the unknown function is expressed under one or more integral signs. A linear integral is shown by this equation

$$h(s)g(s) = f(s) + \lambda \int_a K(s, t)g(t)dt \quad (1.1)$$

When there is room to change or hold constant the upper limit for integration. The three functions $h(s)$, $f(s)$, and $K(s, t)$ are known. But first, we need to locate the function $g(s)$. The kernel function K denotes this instrument (s, t) . Depending on the true values of s and t , these functions may take on complicated values.

1.5.1 Integral equation (IE) types

In general, three distinct kinds of integral equations (IE) exist.

(i) Integral equations (IE) of Fredholm

Every Fredholm integral equation (IE) has the same maximum permitted integration term, b . (a) Since the real part of $h(s)$ is zero in the first kind of Fredholm integral equation (IE) (1.1),

$$f(s) + \lambda \int_a^b K(s, t)g(t)dt = 0 \quad (1.2)$$

(b) (a) With $h(s) = 1$, we get the second kind of the Fredholm integral equation 1.1

Thus

$$g(s) = f(s) + \lambda \int_a^b K(s, t) g(t) dt \quad (1.3)$$

(c) In equation (1.3), $f(s)$ is defined as zero, and the Fredholm integral equation is said to be homogeneous and of the second type. Thus

$$g(s) = \lambda \int_a^b K(s, t) g(t) dt \quad (1.4)$$

(ii) Volterra Integral equations (IE)

The third, fourth, and homogeneous type Volterra integral equations (IE) all use the same formula, , where $b = s$ denotes the upper limit of integration.

(iii) Equations for the Singular Integral

When the integration range reaches an infinite point or the kernel becomes infinite, this one-of-a-kind integral equation (IE) characterises the integral equation. One kind of such equation is the integral.

$$g(s) = f(s) + \lambda \int_{-\infty}^{\infty} e^{-|s-t|} g(t) dt \quad (1.5)$$

And

$$f(s) = \int_0^s \left[1/(s-t)^\alpha \right] g(t) dt \quad 0 < \alpha < 1 \quad (1.6)$$

Are integral equations (IE) that only have one solution

1.5.2 Integral equations (IE) of Other Types

A number of varieties of integral equations, such as dual, triple, and others, throughout the past half-century that have aided mathematical physics in solving mixed boundary value issues. We have distilled the material on certain forms of integral equation (IE) development into this brief summary.

1.5.3 Dual Integral equations (IE)

$$\int_0^{\infty} w(u) \phi(u) K(xu) du = f(x), \quad 0 < x < 1 \quad (1.7)$$

$$\int_0^{\infty} \phi(u) K(xu) du = g(x), \quad x > 1 \quad (1.8)$$

Let $w(u)$ be any function of u where u is the weight function and $K(x, u)$ is the kernel of the two integral equations. The functions $f(x)$ and $g(x)$ are prescribed (known) functions, and the function $\phi(u)$ is an unknown function to be determined? $J_{\mu}(xu)$ is an unknown function to be determined? In practice, the kernels tend to be of either Bessel or trigonometric type, $\sin(xu)$, $\cos(xu)$ etc.

1.5.4 Methods of Solutions

The answer to a dual integral issue may be found using any one of a number of available approaches. A few of the approaches are as follows:

(i) Mellin Transform Application

Applying the Mellin transform technique has greatly improved the theory of dual integral equations. A pioneering work that dealt with these twin integral problems head-on was (Titchmarsh, 1937).

$$\int_0^{\infty} u^{\alpha} \phi(u) J_{\mu}(xu) du = f(x), \quad 0 < x < 1 \quad (1.9)$$

$$\int_0^{\infty} \phi(u) J_{\mu}(xu) du = g(x), \quad x > 1 \quad (1.10)$$

Under the condition that $g(x) = 0$. The Wiener-Hopf methodology [128] is closely connected to his method of solution, and his solution is valid for $\alpha > 0$.

The Mellin transform method has significantly contributed to the enhancement of dual integrated equation theory. One of the earlier citations to explicitly use the Mellin transform for resolving dual integral issues was Titchmarsh in 1937.

By using the theory of Mellin transforms but differently than that of (Titchmarsh, 1937), (Williams, 1961) found out the solution of the equations [1.9] and (1.10). His solution is valid for $-2 < \alpha < 0$ and $0 < \alpha < 2$ respectively.

By using the Busbridge's form of the solution (Noble, 1958) obtained the solution of equations (1.9) and (1.10) valid for $-2 < \alpha < 0$. He also gave the method to extend the range of α .

Mandal and Mandal (1995) recently shown that dual integral equations (IE) may be solved using inverse Weber-Orr transformations.

(ii) Use of Integral Representation Method

It is possible to solve many dual integral equations (IE) with relative ease by using this method. Applying integral representations to the unknown functions, the solutions to the equations (1.9) and (1.10) under different conditions have been discovered by (Tranter, 1951) and (Das., B.R., 1969). In (Cooke, 1956), using Tranter's method, the equations (1.9) and (1.10) were reduced to second-kind Fredholm integral equations (IE) with $g(x) = 0$. Lebdev and Uflyand (1958) impacted the equations' form.

$$\int_0^{\infty} [1 - H(u)] \phi(u) J_0(xu) du = f(x), \quad 0 < x < a \quad (1.11)$$

$$\int_0^{\infty} \phi(u) J_0(xu) du = 0, \quad x > a \quad (1.12)$$

Additionally, the solution was found to be a set of symmetric kernel Fredholm integral equations.

Srivastava and Parihar (1968) recently used this approach to solve dual integral equations requiring inverse Mellin transformations. Several other scholars have used this method to resolve other equations; they include (Burlak, 1962), (Copson, 1961), (Sneddon, 1960), (Lowengrub and Sneddon, 1962), (Szefer & Srivastava, 1964), and (Westmann, 1965). In 1986, Nasim discovered the solution to two integral equations by use of Weber-Orr transformations. A polynomial population of dual integral equations was presented by Rahman (1967).

(iii) Employing Operators for Fractional Integration

Fractional integral operators were proposed by Erdelyi and Kober in 1953, which may be used with a certain class of dual integral equations. Later on, in the field of applied mathematics, further varieties for dual integral equations (IE) (IE) emerged. Sneddon (1966) and Erdelyi and Sneddon (1957) later developed these techniques further, and refined the operator techniques. Several scholars continued to develop and improve the theory, including Ahuja (1973), Erdélyi (1954), Fox (1961), Jageta (1974), Kesharwani (1978), Saxena (1980, 1982), and Tanno (1987). This academic community investigated dual integral equations (IE) with the help of special functions like Bessel functions and inverse Mellin transformations. The method's use was further expanded when Saxena & Sethi (1985) & Sethi and Banerji (1986) investigated dual integral equations (IE) via Meijer's G-functions and H-functions. Saxena and Kushwaha (1984) proved how integrated fractional operators were efficient and resilient when dealing with generalised integral equations by using this approach to solve dual integral equations (IE) using the H-function.

(iv) An Approach Based on Function Theory

Using a function theoretic approach, Schmeltzer and Lewin (1966) resolved two integral equations (IE) using a Hankel kernel. In order to handle a broader class for dual integral equations (IE) which arise in fluid mechanics, Bhattacharya (1971) expanded the approach.

(v) Use of Weber Sonine-Schafheitlin Integrals

When solving a variety of dual integral problems, Weber-Sonine-Schafheitlin integrals are used. As shown by Gordon (1958), the system with equations (1.9) and (1.10) may be solved using a discontinuous integral that combines Bessel functions with Sonine's integral (1894).

Peters (1974) found the solution for more general equations.

$$\int_0^{\infty} u^{\alpha} \phi(u) J_{\mu}(xu) du = f(x), \quad 0 < x < 1 \quad (1.13)$$

$$\int_0^{\infty} \phi(u) J_{\nu}(xu) du = g(x), \quad x > 1 \quad (1.14)$$

Utilizing Sonine's integrals, he was able to get the solutions to these problems in a very straightforward manner. Additionally, he pondered

$$\int_k^{\infty} u^{\alpha} \phi(u) J_{\mu}(xu) du = f(x), \quad 0 < x < 1 \quad (1.15)$$

$$\int_0^{\infty} (u^2 - k^2)^{\beta} \phi(u) J_{\nu}(xu) du = g(x), \quad x > 1 \quad (1.16)$$

Since, under various circumstances, $k \leq 0$ Ahiezer examined several specific instances of the aforementioned equations. Kumar took diffraction theory equations into account as well.

(vi) Application of Generalised Functions Method

Srivastava and Parihar established a technique for solving DIE using generalised functions in 1968 and gave answers to some of these issues. Srivastava (1972) recently examined tempered distributions and dual integral equations (IE) using trigonometric kernels. His L2-theory may hold the key to solving dual trigonometric problems (Srivastava, 1976). Two integral equations (IE) were solved by Ahuja (1984) using generalized functions. Distributional Watson transformations were used by Pathak (1985) to solve dual equations.

(vii) Use of Approximate Methods

King (1936) proposed an approximation approach to solve, with any degree of precision, duality integral equations (IE) of the form (1.9) and (1.10) while considering an acoustical concern. As an alternative method for estimating solutions to dual integral equations, Noble (1958) reduced the two equations to one and showed that it is similar to a variational principle. To numerically solve certain dual integral issues, Linz (1969) proposed an alternative method. Asymptotic solutions to a class of dual integral equations (IE) were shown by Aizikovich (1990).

(viii) Method for Trigonometric Kernels

The approximation approach proposed by King (1936) in response to an acoustical issue, according to him, might be utilised to deal with dual integral equations (IE) of types (1.9) and (1.10) to any desired degree of precision. By combining into a single integral equation (IE) & showing that it was the same as a variational principle, Noble (1958) provided an alternate way of approximating ways to solve dual integral equations. Linz suggested an alternative numerical approach to address certain dual integral problems (1969). In 1990 Aizikovich found asymptotic solutions to a system of dual integral equations.

(ix) Use of Multiplying Factor Method

The original suggestion for this method for addressing equations (1.9) & (1.10) came from Noble in 1963. He clarified the connection to a Wiener-Hopf algorithm even

further. Dwivedi (1966) & Jageta (1968) were among those who used this approach to resolve multiple equations. Pathak (1985) used more generalised H-functions to expand the method to dual integral equations. Multiple solutions were conceivable for the dual integral equations (IE) which Banerjee & Kar (2000) used.

1.5.5 DIE (Dual integral equations) simultaneously

Concurrent identification of events (DIE) of the kind

$$\int_0^{\infty} \sum_{j=1}^n c_{ij}(x) \phi_j(x) J_{u_i}(xy) dx = p_i(y), \quad y \in I_1 \quad (1.17)$$

$$\int_0^{\infty} \phi_j(x) J_{u_i}(xy) dx = 0, \quad y \in I_2 \quad (1.18)$$

In 1964, Erdogan and Bahar looked at the case where we have $i=1, 2, \dots$, and $n_i=1, 2, \dots, n$. In his own work, Westmann (1965) dealt with the special case of $n=2$. The use of trigonometric and Bessel kernels in concurrent dual integral equations (IE) was further explored by Erdogan (1968). Some more scholars who have investigated certain simultaneous dual integral equations (IE) include Dwivedi (1970) and Khadem (1971). Fan examined many systems with dual equations (1979).

Subsequently, via fractional integration operators, that are extensions of the aforementioned Erdélyi-Kober operators, Saxena (1967), together with Dwivedi and Sharma (1976), formalised the answer to certain simultaneous differential integral equations (IE) utilising H-functions as kernels. Additionally, they formalised solutions for simultaneous dual integral equations (IE) using H-functions of n variables (Dwivedi and Sharma, 1977). Dwivedi and Singh (1977) examined convolution-type simultaneous dual integral equations.

1.5.6 Triple Integral Equations

It is necessary to solve the exact duality integral equations (IE) (1.9) as well as (1.10) in order to fix an auditory problem, King (1936) proposed an approximation approach. The following methods may be used to find solutions to integral equations (IE) with three variables.

(i) The Dual Series Method

Tranter One of the first professors to take an interest in three-dimensional integral equations (IE) (1960). The following equations had their answers sent to him:

$$\int_0^{\infty} \phi(u) J_v(xu) du = \begin{cases} f(x), & 0 < x < a \\ h(x), & b < x < \infty \end{cases} \quad (1.19)$$

$$\int_0^{\infty} u^{\alpha} \phi(u) J_v(xu) du = g(x), \quad a < x < b \quad (1.20)$$

Having transformed the initial set of equations into dual series equations using Jaboci polynomials as their kernels, and with $h(x) = 0$. The aforementioned set was reduced to Fredholm integral equations by Lowndes.

Cooke (20) considered the above equations first with $f(x) = h(x) = 0$ and later extended the method for $f(x) \neq h(x) \neq 0$.

(ii) Utilising Operators for Fractional Integration

Solving triple integral problems of the Titchmarsh type was of great interest to Cooke (1965). To find solutions to the broad class of triple integral equations, he used the Erdélyi-Kober operators and their extended versions (Erdélyi and Kober, 1940). He used Fredholm integral equations (IE) to represent his answers, which contained other known

solutions as special instances. In order to solve triple integral equations (IE) and their corresponding triple series, Cooke (1972) drew a parallel between these equations as well as triple Fourier-Bessel series equations. When confronting related Legendre functions that take imaginary inputs, In order to resolve triple integral problems, Saxena & Sethi (1975) employed a combination of fractional integral operators.

(iii) Inverse Mellin Transform Method

In 1968, Srivastava and Parihar arrived at solutions for the corresponding triple integral equations.

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s\phi(s)p^{-s} ds = f_1(\rho), \quad 0 < \rho < a$$

$$= f_3(\rho), \quad 1 < \rho < \infty$$
(1.21)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tan(\alpha s)\phi(s)p^{-s} ds = f_2(\rho), \quad a < \rho < 1$$
(1.22)

The equations are reduced to dual cosine sequence using Tranter's method, Lowndes (1971) solved three integral equations (IE) that built on Cooke's develop fractional integration operators:

$$M^{-1} \left[\frac{\Gamma(\xi + s/\delta)}{\Gamma(\xi + \beta + s/\delta)} \phi(s); x \right] = 0 \quad \begin{array}{l} 0 < x < a \\ b < x < \infty \end{array}$$
(1.23)

$$M^{-1} \left[\frac{\Gamma(\xi + \eta - s/\sigma)}{\Gamma(1 + \eta + \alpha - s/\sigma)} \phi(s); x \right] = f_2(x), \quad a < x < b$$
(1.24)

Where $\alpha, \beta, \eta > 0, \sigma < 0$ are real parameters. Expanding on Cooke's prior research on operators for fractional integration, Lowndes (1971) resolved the following three integral equations:

$$\int_0^{\infty} \phi(u) J_{2p}(ux) dx = 0, \quad 0 \leq x < a; b < x < \infty \quad (1.25)$$

$$\int_0^{\infty} u^{-2n} \phi(u) J_{2q}(ux) dx = F(x), \quad a < x < b \quad (1.26)$$

Through the aforementioned modifications. Furthermore, Singh (1972) investigated a large number of inverted Mellin style triple integral equations (IE), Chakraborty (1971), and Tweed (1973). On restricted parameter triple integral equations, Tweed (1973) studied inverse Mellin transformations for the kind.

$$M_R^{-1} [s^{-1} A(s), x] = 0 \quad \left. \begin{array}{l} 0 < x < a \\ b < x < \infty \end{array} \right\} \quad (1.27)$$

$$M_R^{-1} [A(s) \cot(\pi s), x] = f(x), \quad a < x < b \quad (1.28)$$

Where the finite Mellin transform M_R is defined by the equation

$$M_R [f(x), s] = \int_0^R [x^{s-1} + R^{2s} x^{-s-1}] f(x) dx \quad (1.29)$$

Dwivedi et al. (2000, 2001) recently solved triple integral problems using inverse Mellin transformations.

(iv) The Finite Hilbert Transform Technique

Researchers Srivastava and Lowengrub have solved a system of three integral equations (IE) using trigonometric kernels (1970). Here was the appearance of their mathematical formulas:

$$F[A(u), y] = 0 \quad 0 < y < a; b < x < \infty \quad (1.30)$$

$$F[u^p A(u), y] = f(y) \quad a < y < b \quad (1.31)$$

The Fourier sine or cosine transform, denoted by F and represented by $p+1$, occurs here. By describing $A(u)$ by an appropriate function, the previously stated equations dropped to a well-known Hilbert problem. Equation (1.31) ended up being resolved by simplifying the set into a Hilbert problem, which allowed us to get the answer for the previously mentioned set with an additional factor $(1+H(u))$.

Singh (1972) correctly solved equations (1.30) & (1.31) using the Hilbert transform approach. In 1970, Spence found a Wiener-Hopf solution to certain triple integral equations (IE) for an electrified disc in a coplanar gap that include Hankel transformations (1970). Several fracture concerns in elasticity were handled by Tweed and Langmuir (1974) using inverse Mellin transformations and different triple integral equations.

1.5.7 Concurrent Triple Integral Equations

Physical situations using concurrent triple integral equations (IE) have received very little attention from academic publications. The first to resolve simultaneous triple integral formulae utilising H-functions using the fractional integration operator's technique was Dwivedi and Sharma (1977). Moreover, a simultaneous system for three integral equations (IE) including H-functions was formally resolved by Dwivedi and Singh (1978). There are several factors to consider. Solutions to many triple integral equations (IE) are given in chapter three of this thesis by using Legendre functions as kernels in the inversion produced by the extended Mehler-Fock transform and generalised Legendre function properties.

1.5.8 Quadruple Integral Equations

For mixed boundary value issues, quadruple integral equations have been used. Ahmad was the pioneer in bringing us these particular quadruple integral equations:

$$\int_0^{\infty} t^{-2\alpha} \phi(t) J_{\mu}(xt) dt = f_1(x), \quad 0 < x < a$$

$$= f_3(x), \quad b < x < c$$
(1.32)

$$\int_0^{\infty} t^{-2\beta} \phi(t) J_{\mu}(xt) dt = G_2(x), \quad a < x < b$$

$$= G_4(x), \quad c < x < \infty$$
(1.33)

Where (x) is a function that has to be found and f_1, f_2, f_3 , and f_4 are specified functions of x . The above equations were reduced to second class Fredholm integral equations with the use of Erdelyi-Kober operators.

Dange and Singh, (1972) obtained the formal solution of four equations

$$\int_0^{\infty} \phi(t) P_{-\frac{1}{2}+it}(\cosh x) dt = f_1(x), \quad 0 < x < a$$

$$= f_3(x), \quad b < x < c$$
(1.34)

$$\int_0^{\infty} t \tanh(\pi t) \phi(t) P_{-\frac{1}{2}+it}(\cosh x) dt = f_2(x), \quad 0 < x < a$$

$$= f_4(x), \quad c < x < \infty$$
(1.35)

Where $\phi(t)$ the unknown function has to be discovered so it can be numerically solved using the Fredholm integral equation.

Cooke (1972) used the Erdelyi-Kober, Sneddon and Srivastava operators to solve quadruple integral equations that contained Bessel functions with arbitrary weight functions. These equations were transformed into Fredholm integral equations by him. Dwivedi and Trivedi (1979) applied Fredholm integral equations to get solutions of equations that were more general than the equations that Cooke studied.

Also, several sets of quadruple integral equations have been studied and solutions to them have been found by other authors such as Gupta and Chaturvedi (1970), Saxena and Sethi (1972), Jain and Singh (1976) and Singh (1974).

1.5.9 Six Integral Equations

“It is also possible to solve mixed boundary value issues using the six integral equations. The solution of six integral equations (IE) using the extended Legendre function as the kernel was covered in Chapter 4 of the thesis. Numerical techniques are the most effective means of obtaining the answer, which is expressed as the Fredholm integral equation.”

It is time to move on to some elasticity problems using integral equations.

1.6 ISSUES WITH ELASTICITY

- Fracture issues with incompatible medium.
- Thermal issues with elasticity.

1.6.1 Crack Issues in Diverse Media

Several issues involving different types of elastic media that may be addressed using integral equation (IE) techniques are now outlined. The joining of materials with varying mechanical characteristics is a common technique in engineering. In order for the system of different materials to transfer loads from one material to another across their interfaces, it must function as a unified whole. If a crack becomes big enough, it might cause failure due to high local stresses induced by flaws or fractures in the material or at the interface. Understanding the stress condition linked to these fissures in the unique material system is, therefore, crucial. There are several peculiarities with the analytic stress solution when the signs of the stresses change rapidly near the interface fracture point.

The radial distance from $r_1^{-1/2} \sin(\text{or } \cos)$ the fracture $\in \log(r_1/a)$ border ($r \geq 1$), the crack size (a), and the bimaterial constant E (which depends on the mechanical characteristics of the bordering materials) create the argument that describes this wildly variable feature. This behavior was first discovered by Williams (1959) while studying two-dimensional fracture issues.

A straight line joining two plates with differing elastic properties provides support for a fracture; this problem has been considered by Sih and Rice (1964). The stress solution shows that critical stresses are localised to a small area around the fracture tip, oscillate, and are negatively proportionate to the square root corresponding to the distance radial to the front when both materials were isotropic & homogenous. Given the critical nature of the stress distribution close to a fracture tip in determining the plate strength, we provide a pair of integral equations (IE) which capture this distribution. Different materials' characteristic equations for the extension & bending of fractured plates also show certain commonalities.

“Considered under this broad topic is the joining of two semi-infinite elastic media with differing characteristics along a plane by a sequence of flat holes constructed like concentric rings; Erdogan (1965) addresses this issue. The issue is characterized as a set of concurrent singular integral equations (IE) by applying Green's functions on the semi-infinite plane. For connected dominating systems with Cauchy kernels, a closed form solution is given. Full understanding of the answer requires familiarity with the steps that transformed the issue from two Fredholm integral equations (IE) to a system of linear algebraic equations. Materials experiencing uniform temperature changes or axial pressures, fissures in the shape of pennies, and bonded media with an axially symmetrical external notch are among the examples he provided for the charting of contact stresses.” A penny-shaped fissure at the interface of two bonded, different materials that were uniformly compressed was investigated by Mossakovskii and Rybka in 1964. They employed the Muskhelishvili method to analyze the axisymmetric elasticity problem that was brought down to a pair of analytic plane-based functions. Out of the complicated potentials, the physical quantities in the material were obtained through the relevant transformations. Based on fracture mechanics theories, stress singularities along the crack edge, and stress intensity parameters are essential in predicting when a material is going to collapse.

Two mixed boundary value problems that were axisymmetric were studied by Arin and Erdogan in 1971. In the former, a layer of elastic material was placed in-between two half-spaces, which were semi-infinite and had a central crack and a penny-shaped crack running parallel to the interfaces; the two half-spaces were assumed to have the same

elastic properties. The latter considered the fact that each of the three different materials could have a fracture. The parameters of stress intensity parameters were analysed in the study in relation to the ratio of the layer thickness to the crack radius and the numerical results are presented. The stress and displacement fields around a Griffith fracture at the interface of two different elastic half-planes were studied by Lowengrub and Sneddon (1973). The dual integral equations were reduced to a Riemann boundary value problem by using Fourier transformations, which allowed them to reframe the issue and get a closed-form solution. The Griffith fractures were also experimented in further research at interfaces involving elastic half-planes and stiff bases.

Lowengrub and Sneddon extended the study of a penny sized fissure at the interface of two elastic half-spaces bonded and under constant pressure in 1974. They employed the methods of integral equations and Hankel transformations to characterise displacement fields. Abel transformations and complex representation of functions simplified the issue to a single integral equation. The technique made formulas of contact stresses easier and made fracture energy easy to measure. In addition, Lowengrub (1975) studied the stress fields at the interface between different elastic half-planes as related to two Griffith fractures. We reduced complex integral problems by the application of Fourier transform. These experiments established that these methods of analysis could be applied to arrangements where there were many coplanar cracks at the boundaries between materials as well.

Also, Srivastava, Palaiya, as well as Choudhary (1979) looked into systems of multiple integral equations (IE) for mythical and hyperbolic functions. A general answer to the issue of a Griffith crack which is at the junction of two joined pieces of different stiffness was found by Srivastava, Choudhary, and Palaiya in 1979. They used inverse Mellin transform methods to do this. Later, Wu, Yang, as well as Fan (2002) looked into how a shifting Griffith crack behaves at the point where two connected, different elastic half-planes meet.

Bassani and Erdogan (1979) examined this antiplane shear problem, which involved two related, but distinct elastic half-planes, having a semi-infinite fracture or with two arbitrarily positioned collinear fractures. This was solved using a concentrated wedge

load to enable us to establish the angle of stress distribution and factor of stress intensity of the semi-infinite fracture. Infinite fractures had only two equations that were integrable. In completely immersed fractures in a homogenous medium, where the interface was contacted by only one crack tip, and those cracks intersecting the contact at other angles, a set of numbers was discovered.

Barenblatt fracture in an bonded-elastic half-plane contact was investigated by Parihar and Garg (1976). It may seem that the fact of collinear fractures form at the junction of two connected elastic half-planes is an extension of the issue, however Parihar & Garg (1973) addressed this particular instance.

(iii) Thermal Elasticity Issues

(a) Stresses caused by heat in homogeneous medium

The evidence on the distribution of stress around elastic body cracks namely Griffith, penny-shaped and Barenblatt cracks has received a great deal of attention due to the critical role it plays in fracture mechanics. Concurrently, there has been a rise in the amount of work devoted to studying thermal stresses close to elastic material fissures. Olesiak and Sneddon (1960) studied thermal stress fields in an infinite solid having a penny-shaped fracture. This was followed by Florence and Goodier (1963) who studied a linear thermoelastic problem in which a uniformly heated insulated penny-like crack was studied. Infinite solid solutions were used to obtain solutions to the infinite solid, which were typically written in closed form, and which were obtained by methods based on dual integral equations.

A range study of thermal stresses distribution was conducted by Das (1969) on a long elastic cylinder with a concentric penny-shaped fracture. In a similar work, the distribution of stress in an elastic material having a penny-shaped fracture perpendicular to a free surface was also studied. The issue of mixed boundary values of thermal stress of a long circular cylinder with a penny-shaped fracture that is symmetrically located under specified conditions of temperature was earlier discussed by Das (1968).

The key issues of the Kassir and Sih (1969) study of the linear thermoelastic problems in an infinite elastic media were heat stress and displacement fields. In their study,

Srivastava, Kumar, and Nath (1972) zeroed attention on the thermal stresses that an infinite body experiences in a study of isotropic and homogeneous materials, under a linear thermoelasticity theory. They extended their study many years later in 1974 where they looked into the distribution of heat stresses in an indefinitely long, homogeneous, isotropic elastic cylinder with two longitudinally running fractures.

(b) Stresses caused by heat in different mediums

In 1972, Srivastava, Kumar, and Nath investigated the linear equations of thermoelasticity equilibrium of isotropic and homogeneous materials. They considered the distribution of thermal stresses in an infinite solid, having a cylindrical hollow in it, and by which are two parallel strip fractures along the length of the hollow. A later study (1974) built upon their previous work and investigated the distribution of heat stresses in an infinitely long, uniform, two-fracture infinitely long isotropic elastic cylinder. Simplified analysis of quasi-static thermal stresses in bonded dissimilar materials with planar interface fractures, which may be completely or partly insulated, was suggested by Brown and Erdogan (1968). Bregman and Kassir (1974) further studied the thermal fracture behaviour of bonded dissimilar materials with a penny shape crack. Srivastava, Palaiya and Choudhary (1977) studied the distribution of thermal stress in an elastic layer with penny-shaped fracture connected between two different half-spaces and oriented perpendicular to the interfaces. They continued to examine thermal stresses in linked different elastic half-spaces with a penny-shaped interface fracture in a 1978 follow-up research.

1.7 DUAL INTEGRAL EQUATIONS SIMULTANEOUS

Not long ago, Masood and Srivastava solved the dual integral equations (IE) involving two-variable G-functions with fractional integral operators (1970). This chapter focuses on solving simultaneous integral equations based on two-variable G-functions. Here is how G-functions were defined by Agrawal (1965):

$$G_{p+n, (t+v_1, t+v_2), s, (q+m_1, q+m_2)}^{n, v_1, v_2, m_1, m_2} \left| \begin{matrix} (\epsilon_n + n) \\ (\gamma_{t+v_1}) (\gamma'_{t+v_2}) \\ (\delta_s) \\ (\beta_{q+m_1}) (\beta'_{q+m_2}) \end{matrix} \right| \quad (1.36)$$

$$= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m_1} \pi \Gamma(\beta_j - \xi) \prod_{j=1}^{v_1} \pi \Gamma(\gamma_j + \xi) \prod_{j=1}^{m_2} \pi \Gamma(\beta_j - \eta) \prod_{j=1}^{v_2} \pi \Gamma(\gamma'_j + \eta)}{\prod_{j=1}^q \pi \Gamma(1 - \beta_{m_1+j} + \xi) \prod_{j=1}^t \pi \Gamma(1 - \gamma_{v_1+j} - \xi) \prod_{j=1}^q \pi \Gamma(1 - \beta'_{m_2+j} + \eta)} \\ \frac{\prod_{j=1}^n \pi \Gamma(1 - \epsilon_j + \xi + n) x^n}{\prod_{j=1}^t \pi \Gamma(1 - \gamma'_{2+j} - \eta) \prod_{j=1}^p \pi \Gamma(\epsilon_{n+j} - \xi - n) \prod_{j=1}^s \pi \Gamma(\delta_j + \xi + n)} d\xi dn$$

where $0 \leq m_1 \leq q$, $0 \leq m_2 \leq q$, $0 \leq v_1 \leq t$, $0 \leq v_2 \leq t$, $0 \leq n \leq p$ and $(a_r) = a_1, a_2, \dots,$

$(a_{r+1}, s) = a_{r+1}, a_{r+2}, \dots, a_s$.

(1.137)

The sequence of parameters $(\beta_{m_1}), (\beta'_{m_1}), (\gamma_{v_1}), (\gamma'_{v_1})$ and (ϵ_n) do not correspond with any of the integer's poles. Any required indentation of the integration routes is done in such a way that all poles of $\Gamma(\beta_j - \xi)$, $j = 1, \dots, m_1$ and $\Gamma(\beta'_j - \eta)$, $k = 1, \dots, m_2$ lie to the right and those of $\Gamma(\gamma_j + \xi)$, $j = 1, \dots, v_1$, $\Gamma(\gamma'_j + \eta)$, $k = 1, \dots$ and $\Gamma(1 - \epsilon_j + \xi + n)$, $j = 1, \dots, n$ are located on the imaginary axis's left side.

The integral (2.1) converges if

$$\left. \begin{aligned} p+q+s+t &< 2(m_1 + v_1 + n) \\ p+q+s+t &< 2(m_2 + v_2 + n) \\ \text{and } \arg x &< \pi \left[m_1 + v_1 + n - \frac{1}{2}(p+q+s+t) \right] \\ \arg y &< \pi \left[m_2 + v_2 + n - \frac{1}{2}(p+q+s+t) \right] \end{aligned} \right| \quad (1.38)$$

1.8 DOUBLE INTEGRAL EQUATIONS

The dual integral equations that hold at the same time will be covered in the following:

$$\int_0^\infty \int_0^\infty G_{p+n, (t+v_1, t+v_2), 0, (q+m_1, q+m_2)}^{n, v_1, v_2, m_1, m_2} \left[\begin{array}{c} xu \\ yu' \end{array} \left| \begin{array}{c} (\varepsilon_{p+n}) \\ (\gamma_{v_1}^\theta), (\gamma_{v_2}^\theta) \\ (\beta_{m_1}), (\beta_{m_2}') \\ (\gamma_{v_1+t}), (\gamma_{v_2+t}') \\ (\beta_{m_1+q}^\theta), (\beta_{m_2+q}') \end{array} \right. \right] \cdot$$

$$\sum_{\psi=1}^{n'} a_{\psi\theta} f_\psi(u, u') du du' = \theta_{1\theta}(x, y), \quad 0 < x < 1, 0 < y < 1$$
(1.39)

$$\int_0^\infty \int_0^\infty G_{p+n, (t+v_1, t+v_2), 0, (q+m_1, q+m_2)}^{n, v_1, v_2, m_1, m_2} \left[\begin{array}{c} xu \\ yu' \end{array} \left| \begin{array}{c} (\varepsilon_{p+n}) \\ (c), (c') \\ (b_{m_1}^\theta), (\beta_{m_2}^\theta) \\ (c_{+t}^\theta), (c_{+t}') \\ (b_{m_1+q}^\theta), (b_{m_2+q}') \end{array} \right. \right] \cdot$$

$$\sum_{\psi=1}^{n'} b_{\psi\theta} f_\psi(u, u') du du' = \phi_{2\theta}(x, y), \quad x > 1, y > 1$$

$$\theta = 1, 2, \dots, n. \quad (1.40)$$

where $a_{\psi\theta}$, $b_{\psi\theta}$ are constants, $\phi_{1\theta}(x, y)$, $\phi_{2\theta}(x, y)$ are given and $f_\psi(x, y)$ is to be determined.

1.9 THE TRANSFORMATION OF MELLIN

Here, we use the Mellin kind of double integral, assuming x and y are positive.

$$g_{\psi}(x, y) = \frac{1}{(2\pi i)^2} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\sigma'-i\infty}^{\sigma'+i\infty} x^{-r} y^{-s} f_{\psi}(r, s) dr ds$$

then

$$(1.41)$$

$$f_{\psi}(r, s) = \int_0^{\infty} \int_0^{\infty} x^{r-1} y^{s-1} g_{\psi}(x, y) dx dy$$

$$(1.42)$$

under appropriate circumstances because of Reed [13–7].

We might write (1.41) and (1.42) as

if

$$M[f_{\psi}(x, y)] = F_{\psi}(r, s) = \int_0^{\infty} \int_0^{\infty} x^{r-1} y^{s-1} g_{\psi}(x, y) dx dy$$

then

$$(1.43)$$

$$M^{-1}[F_{\psi}(r, s)] = g_{\psi}(x, y) = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} x^{-r} y^{-s} F_{\psi}(r, s) dr ds$$

$$(1.44)$$

We prove a theorem about the two-variable Mellin transform, which is equivalent to Fox's Parseval theorem for one variable.

If

$$M[h(u, u')] = H(r, s),$$

$$\text{and } M[F_{\psi}(xu, yu')] = x^{-r} y^{-s} F_{\psi}(r, s)$$

$$(1.45)$$

Where

$$M[f_\psi(u, u')] = F_\psi(r, s)$$

then

$$\int_0^\infty \int_0^\infty h(xu', yu') f_\psi(u, u') = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} x^{-r} y^{-s} H(r, s) F_\psi(1-r, 1-s) dr ds \quad (1.46)$$

It is evident from equations (1.44) that

$$M \left[G_{p+n, (t+v_1, t+v_2), 0, (q+m_1, q+m_2)}^{n, v_1, v_2, m_1, m_2} \begin{matrix} x \\ y \end{matrix} \left(\begin{matrix} \epsilon_{p+n} \\ \gamma^\theta \\ \beta_{m_1} \\ \gamma_{t+1} \\ \beta_{m_1+q}^\theta \end{matrix} \right) \left(\begin{matrix} \gamma'^\theta \\ \beta'_{m_2} \\ \gamma'_{t+1} \\ \beta'_{m_2+q}^\theta \end{matrix} \right) \right] = \frac{\prod_{j=1}^{m_1} \pi \Gamma(\beta_j + r) \prod_{j=1}^{v_1} \pi \Gamma(\gamma_j^\theta - r) \prod_{j=1}^{m_2} \pi \Gamma(\beta'_j + s) \prod_{j=1}^{v_2} \pi \Gamma(\gamma'_j{}^\theta - s) \prod_{j=1}^n \pi \Gamma(1 - \epsilon_j - r - s)}{\prod_{j=1}^q \pi \Gamma(1 - \beta_{m_1+j}^\theta - r) \prod_{j=1}^t \pi \Gamma(1 - \gamma_{v_1+j} + r) \prod_{j=1}^q \pi \Gamma(1 - \beta'_{m_2+j} - s) \prod_{j=1}^t \pi \Gamma(1 - \beta'_{v_2+j} + s)} \quad (1.47)$$

$$\frac{1}{\prod_{j=1}^p \pi \Gamma(\epsilon_{n+j} + r + s)} = \psi(r, s) \quad (1.48)$$

One using $M[f_\psi(u, u')] = F_\psi(r, s)$ and applying (1.46) to (1.39) and (1.40), we find that

$$\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \chi(r, s) x^{-r} y^{-s} \sum_{\psi=1}^{n'} \alpha_{\psi} \psi \phi F_\psi(1-r, 1-s) dr ds = \phi_{10}(x, y) \quad 0 < x < 1, 0 < y < 1 \quad (1.49)$$

$$\begin{aligned}
& \frac{1}{(2\pi i)^2} \int_{-\infty}^{i\infty} \int_{-\infty}^{i\infty} \chi(r, s) x^{-r} y^{-s} \sum_{\psi=1}^{n'} F_{\psi}(1-r, 1-s) dr ds \\
& = \phi_{2\theta}(x, y) \quad x > 1, y > 1
\end{aligned} \tag{1.50}$$

where $\theta = 1, 2, \dots, n$,

$$\begin{aligned}
\bar{\chi}(r, s) = & \frac{\prod_{j=1}^{m_1} \Gamma(b_j^{\theta} + r) \prod_{j=1}^{v_1} \Gamma(c_j - r) \prod_{j=1}^{m_2} \Gamma(b_j^{\theta} + s) \prod_{j=1}^{v_2} \Gamma(c_j' - s)}{\prod_{j=1}^q \Gamma(1 - b_{m_1+j} - r) \prod_{j=1}^t \Gamma(1 - c_{v_1+j}^{\theta} + r) \prod_{j=1}^q \Gamma(1 - b_{m_2+j}' - s)} \\
& \frac{\prod_{j=1}^n \Gamma(1 - \varepsilon_j - r - s)}{\prod_{j=1}^t \Gamma(1 - c_{v_2+j}^{\theta} + s) \prod_{j=1}^p \Gamma(\varepsilon_{n+j} + r + s)}
\end{aligned} \tag{1.51}$$

And $x(r, s)$ is given by (1.48).

1.10 SOLVING DUAL INTEGRAL EQUATIONS (IE) SIMULTANEOUSLY

In this case, we transform equations (1.39) and (1.40) into two equations that share a kernel by using fractional integration operators of two variables, similar to the ones Erdélyi (1950-1951) formulated for one variable.

$$\begin{aligned}
\tau_{1,1}[\alpha, \beta, \alpha', \beta' : r, s : w(x, y)] &= [rs/\Gamma(\alpha)\Gamma(\alpha')] x^{-r\alpha+r-\beta-1} y^{-s\alpha'+s-\beta'-1} \\
& \int_0^x \int_0^y (x^r - v^r)^{\alpha-1} (y^s - v'^s)^{\alpha-1} (y^s - v'^s)^{\alpha'-1} v^{\beta} v'^{\beta} w(v, v') dv dv'
\end{aligned} \tag{1.52}$$

$$\begin{aligned}
R_{1,1}[\alpha, \beta, \alpha', \beta' : r, s : w(x, y)] &= [rs/\Gamma(\alpha)\Gamma(\alpha')] x'^{\beta} y'^{\beta'} \int_x^{\infty} \int_y^{\infty} (v^r - x^r)^{\alpha-1} \\
& (v'^s - y'^s)^{\alpha'-1} v^{-\beta-r\alpha+r-1} v'^{-\beta-s\alpha'+s-1} w(v, v') dv dv'
\end{aligned} \tag{1.53}$$

In the constructed form we write

$$\tau[(\gamma_j - c_j)(\gamma'_j - c'_j)c_j^{-1} : 1, 1 : w(x, y)] = \tau_j[w(x, y)] \quad (1.54)$$

$$\tau(\beta_{m_1+k} - b_{m_1+k})(\beta'_{m_2+k} - b'_{m_2+k}) - \beta'_{m_1+k} - \beta'_{m_2+k} : 1, 1 : w(x, y) \\ \tau_k^{*1}[w(x, y)] \quad (1.55)$$

$$R[(b_1 - \beta_1)(b'_1 - \beta'_1) : \beta_1, \beta'_1 : 1, 1 : w(x, y)] = R_1[w(x, y)] \quad (1.56)$$

$$R[(c_{v_1+h} - \gamma_{v_1+h})(c'_{v_2+h} - \gamma'_{v_2+h})c'_{v_1+h}c'_{v_2+h} : 1, 1 : w(x, y)] \\ = R_h^{*2}[w(x, y)] \quad (1.57)$$

Now in (1.49) replace x and y by v and v' respectively, multiply by $(x-v)^{\gamma_{v_1}-c_{v_1}-1}(y-v')^{\gamma'_{v_2}-c'_{v_2}-1}v^{c_{v_1}-1}v'^{c'_{v_2}-1}$ and integrate through integral sign w, r, t, v and v' from 0 to x and 0 to y respectively, where $0 < x < 1, 0 < y < 1$, we obtain

$$\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j + r) \prod_{j=1}^{v_1} \Gamma(\gamma_j^\theta - r) \prod_{j=1}^{m_2} \Gamma(\beta'_j + s) \prod_{j=1}^{v_2} \Gamma(\gamma'_j{}^\theta - s)}{\prod_{j=1}^q \Gamma(1 - \beta_{m_1+j} - r) \prod_{j=1}^1 \Gamma(1 - \gamma_{v_1+j} + r) \prod_{j=1}^q \Gamma(1 - \beta'_{m_2+j} - s)} \\ \frac{\Gamma(c_{v_1} - s) \Gamma(c_{v_2} - s) \prod_{j=1}^n \Gamma(1 - \varepsilon_j - s - r)}{\prod_{j=1}^1 \Gamma(1 - \gamma'_{v_2+j} + s) \prod_{j=1}^n \Gamma(\varepsilon_{n+j} + r + s)} x^{-r} y^{-s} \sum_{\psi=1}^{n'} a_\psi \theta F_\psi(1-r, 1-s) dr ds \\ = \frac{x^{1-\gamma_{v_1}^\theta} y^{1-\gamma_{v_2}^\theta}}{(\gamma_{v_1}^\theta - c_{v_1})(\gamma_{v_2}^\theta - c_{v_2})} \int_0^x \int_0^y (x-v)^{\gamma_{v_1}^\theta - c_{v_1} - 1} (y-v')^{\gamma_{v_2}^\theta - c_{v_2} - 1} \\ v^{c_{v_1} - 1} v'^{c_{v_2} - 1} \phi_{10}(v, v') dv dv' = \tau_{v_1, v_2}[\phi_{10}(v, v')] \quad (1.58)$$

where $0 < x < 1$, $0 < t < 1$ by virtue of (1.39) and (1.41)

On transforming (1.45) successively, for $(K = v_1 - 1, \dots, 1; v_2 - 1, \dots, 1)$

by the application of operators $\tau_{k,k}$ and τ_{kk}^{*1} ($k = q, q-1, \dots, 1$) we finally get

$$\begin{aligned} & \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j + r) \prod_{j=1}^{v_1} \Gamma(c_j - r) \prod_{j=1}^{m_2} \Gamma(\beta'_j + s)}{\prod_{j=1}^q \Gamma(1 - b_{m_1+j} - r) \prod_{j=1}^t \Gamma(1 - \gamma_{v_1+j} + r) \prod_{j=1}^q \Gamma(1 - b'_{m_2+j} - s)} \\ & \frac{\prod_{j=1}^{v_2} \Gamma(c'_j - s) \prod_{j=1}^n \Gamma(1 - \varepsilon_j - r - s)}{\prod_{j=1}^t \Gamma(1 - \gamma'_{v_2+j} + s) \prod_{j=1}^p \Gamma(\varepsilon_{n+j} + r + s)} x^{-r} y^{-s} \sum_{\psi=1}^{n'} a_{\psi\theta} F_{\psi}(1-r, 1-s) \\ & dr ds = \tau_{1,1}^{*1} [\tau_{2,2}^{*1} \dots \tau_{q,q}^{*1} \tau_{1,1} \dots \tau_{v_1 v_2} [\phi_{1\theta}(x, y)] \dots] \quad 0 < x < 1, \quad 0 < y < 1 \end{aligned} \quad (1.59)$$

$\theta = 1, 2, \dots, n$.

Where

Likewise, by the use of operators R_1 and R_h^{*2} given in (1.56) and (1.57) respectively, for $l = m_1, m_1 - 1, \dots, 1, m_1, m_2 - 1, \dots, 1$ and $h = t, t - 1, \dots, 1$ It changes into the required shape when subjected to (1.50).

$$\begin{aligned} & \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j + r) \prod_{j=1}^{v_1} \Gamma(c_j - r) \prod_{j=1}^{m_2} \Gamma(\beta'_j + s) \prod_{j=1}^{v_2} \Gamma(c'_j - s)}{\prod_{j=1}^q \Gamma(1 - b_{m_1+j} - r) \prod_{j=1}^t \Gamma(1 - \gamma_{v_1+j} + r) \prod_{j=1}^q \Gamma(1 - b'_{m_2+j} - s)} \\ & \frac{\prod_{j=1}^n \Gamma(1 - \varepsilon_j - r - s)}{\prod_{j=1}^t \Gamma(1 - \gamma'_{v_2+j} + s) \prod_{j=1}^p \Gamma(\varepsilon_{n+j} + r + s)} x^{-r} y^{-s} \sum_{\psi=1}^{n'} a_{\psi\theta} F_{\psi}(1-r, 1-s) \\ & dr ds = \sum_{\psi=1}^{n'} c_{\psi\theta} R_{1,1}^{*2} [R_{2,2}^{*2} \dots R_{t,t}^{*2} R_{1,1} \dots R_{m_1, m_2} [\phi_{2\theta}(x, y)] \dots] \quad x > 1, \quad y > 1 \end{aligned} \quad (1.60)$$

where $c_{\psi\theta}$ are the elements of the matrix $[a_\theta] [b_\theta]^{-1}$, $\theta = 1, 2, \dots, n$.

On setting

$$G_\theta(x, y) = \begin{cases} \tau_{1,1} [\tau_{2,2}^* \dots \tau_{q,q}^* \tau_{1,1} \dots \tau_{v_1 v_2} [\phi_{1\theta}(x, y)] \dots] & 0 < x < 1, 0 < y < 1 \\ \sum_{\psi=1}^{n'} c_{\psi\theta} R_{1,1} [R_{2,2}^* \dots R_{t,t}^* R_{1,1} \dots R_{m_1, m_2} [\phi_{2\theta}(x, y)] \dots] & x > 1, y > 1 \end{cases} \quad (1.61)$$

As they are identical, (1.59) and (1.60) in the left-hand side of the equation denote a common kernel in a compact form.

$$\begin{aligned} & \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j + r) \prod_{j=1}^{v_1} \Gamma(c_j - r) \prod_{j=1}^{m_2} \Gamma(\beta'_j + s) \prod_{j=1}^{v_2} \Gamma(c'_j - s)}{\prod_{j=1}^q \Gamma(1 - b_{m_1+j} - r) \prod_{j=1}^t \Gamma(1 - \gamma_{v_1+j} + r) \prod_{j=1}^q \Gamma(1 - b'_{m_2+j} - s)} \\ & \frac{\prod_{j=1}^n \Gamma(1 - \varepsilon_j - r - s)}{\prod_{j=1}^t \Gamma(1 - \gamma'_{v_2+j} + s) \prod_{j=1}^p \Gamma(\varepsilon_{n+j} + r + s)} x^{-r} y^{-s} \sum_{\psi=1}^{n'} a_{\psi\theta} F_\psi(1 - r, 1 - s) \\ & dr ds = G_\theta(x, y) \end{aligned} \quad (1.62)$$

The two kernel-sharing equations are simplified to a single equation in (1.62). A technique analogous to Fox's one-variable method and an unsymmetrical Fourier transform allow the kernel of (1.62) to be rewritten as (1.62).

$$\begin{aligned} f_\psi(x, y) &= \frac{1}{(2\pi i)^2} \sum_{\psi=1}^{n'} g_{\psi\theta} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^q \Gamma(1 - b_{m_1+j} + r) \prod_{j=1}^t \Gamma(2 - \gamma_{v_1+j} - r)}{\prod_{j=1}^{m_1} \Gamma(1 - \beta_j - r) \prod_{j=1}^{v_2} \Gamma(-1 + c_j + r)} \\ & \frac{\prod_{j=1}^q \Gamma(-b'_{m_2+j} + s) \prod_{j=1}^t \Gamma(2 - \gamma'_{v_1+j} - s) \prod_{j=1}^p \Gamma(2 + \varepsilon_{n+j} - r - s)}{\prod_{j=1}^{m_2} \Gamma(1 - \beta'_j - s) \prod_{j=1}^{v_2} \Gamma(1 + c'_j + s) \prod_{j=1}^n \Gamma(-1 - \varepsilon_j + r + s)} x^{-r} y^{-s} \\ & G'_\theta(1 - r, 1 - s) dr ds \\ & \text{where } M[G_\theta(x, y)] = G'_\theta(r, s). \end{aligned} \quad (1.63)$$

For equations (1.39) and (1.40), this is the formal answer. Applying 1.44 in light of 1.63). The fact that

$$f_{\psi}(x, y) = \sum_{\psi=1}^{n'} g_{\psi\theta} \int_0^{\infty} \int_0^{\infty} G_{p+n, (t+v_1, t+v_2), 0, (q+m_1, q+m_2)}^{p, t, q, q} \left[\begin{matrix} xu \\ yu' \end{matrix} \middle| \begin{matrix} (-1-\varepsilon_{n+1}, p), (-1-\varepsilon_n) \\ (2-\gamma_{+1}, t), (2-c), (2-\gamma'_{+1}, t), (2-c') \\ (-b_{m_1+1}, q), (-\beta_{m_1}), (-b'_{m_2+1}, q), (-\beta'_{m_2}) \end{matrix} \right] G_{\theta}(u, u') du du', \quad (1.64)$$

where $g_{\psi\theta}$ are the elements of the matrix $[a_{\theta}]^{-1}$, $\theta = 1, 2, \dots, n$. When (1.64) written out in full it becomes:

$$f_{\psi}(x, y) = \sum_{\psi=1}^{n'} g_{\psi\theta} \int_0^1 \int_0^1 G_{p+n, (t+v_1, t+v_2), 0, (q+m_1, q+m_2)}^{p, t, q, q} \left[\begin{matrix} xu \\ yu' \end{matrix} \middle| \tau_{1,1}^{\tau_1} [\tau_{2,2}^{\tau_2} \dots \tau_{q,q}^{\tau_q} \tau_{1,1}^{\tau_1} \dots \tau_{v_1 v_2}^{\tau_{v_1 v_2}} [\phi_{1\theta}(u, u')]] du du' \right] + \int_0^{\infty} \int_0^{\infty} G_{p+n, (t+v_1, t+v_2), 0, (q+m_1, q+m_2)}^{p, t, q, q} \left[\begin{matrix} xu \\ yu' \end{matrix} \middle| \sum_{\psi=1}^{n'} c_{\theta} R_{1,1}^{\tau_1^2} [R_{2,2}^{\tau_2^2} \dots R_{t,1}^{\tau_t^2} R_{1,1}^{\tau_1^2} \dots R_{m_1 m_2}^{\tau_{m_1 m_2}^2} [\phi_{2\theta}(u, u')]] du du' \right] \quad (1.65)$$

1.11 REVIEW OF LITERATURE

The potential hypothesis originated from a desire to understand the basic rules of existence. After Lagrange proved that the gravitational force field is a potential field, Green renamed it a potential function. But it is simply prospective, according to Gauss. Potential is a key concept in many mathematical physics issues. Potential emerged as a central concept in Dirichlet, Naumann, and Rabin's electrostatics problems. Analytical function and modern potential theory are closely related. When it comes to analytical functions, the Laplace equation holds. A well-established area of study is analytical functions. So, for issues pertaining to two-dimensional potential theory, the complex variable approach is an effective tool.

The temperature field is in equilibrium with Laplace's equation when dealing with issues of steady-state temperature distribution. Despite writings on the topic by Weber (1873)

and Beltrami (1881), route independent integrals, alternative integral equations, triple relations, or wasn't published until around 1945.

Regarding problems of steady-state distribution of temperatures, the field of temperatures is in balance with Laplace's equation. In spite of writings on the topic by Weber (1873) and Beltrami (1881), the theory of dual series equations, triple relations, route independent integrals, and alternative integral equations (IE) was not tackled until around 1945.

The first concept of crack was proposed in 1913, as stated by C.E. English. Using complex variable analytical functions, N.I. Maskhetishvili offered a much easier answer of the same issue shortly after that in 1918. He used potential theory in his research. The theoretical foundations of crack theory were proposed by Griffith. His main point at the crack tip was molecular cohesion. According to his theory of energy balance, the main way energy is dissipated in brittle materials is via surface energy from fracture faces.

Gdoutos and Sih have established a comprehensive theory of material damage in a body as a result of yielding and/or the beginning and progression of fractures. Gdoutos takes care of the mixed-type problem. The onset and progression of contact issues were examined by Gdoutos and Drakos. It was Laplace and the Fourier transform that these writers used.

1.12 SCOPE OF THE PROPOSED WORK

The normal stress components exhibit singularities at the fracture locations, as we conclude. There is no noticeable movement in the location of the fracture. At the very edges of fissures, plastic zones may form. A separate discussion is necessary for this. The components of displacement and stress may be readily measured at the location (x, y) . Singularities in the stress components are also seen at the square root of the fracture sites that are Cauchy-type in orthotropic materials. The crack opening presents a smooth displacement, much like an isotropic instance. Typically, the plastic zone is more likely to form near the periphery of fractures. Finding out how lengthy an orthotropic material's plastic zone will be will be challenging, but not impossible.

An improved formation of the title problem might be the one with an infinitely continuous isotropic medium with open Griffith fractures and two identical wedges forming an unending row perpendicular to the x-axis. This issue can be viewed as a variant of the one where there is an infinite isotropic medium with an infinite chain of four Griffith-cracks that open up due to the presence of two wedges and infinite chains that superimpose each other at a distance of 2δ on the base row on the x-axis. The cracks at this point are same in size and shape and are parallel to the y-axis as they cross the x-axis.

An indefinitely continuous isotropic material with open Griffith fractures and two identical wedges creating an endless row perpendicular to the x-axis might be a better way to formulate the title issue. This problem, which may be seen as a variation of the original, is based on an unending isotropic material with an endless row of four Griffith-cracks separated by two wedges and an endless number of rows stacking at a 2δ distance from the base row along the x-axis. In this case, the cracks maintain a constant size and form as they cross the x-axis and lay across the y-axis.

1.13 OBJECTIVES

1. To find the stress-intensity factor in some crack problems which are multiply connected domain.
2. To find the stress-intensity factors in the crack problems in orthotropic medium.
3. To find the stress-intensity factors of an elastic strip containing Griffith crack.
4. To solve some crack problems opened by forces at crack faces in a rectangular domain.
5. To determine the distribution of temperature by integral equation (IE) method over multiply connected isotropic infinite medium.
6. To determine the distribution of temperature by integral equation (IE) method over multiply connected orthotropic infinite medium.

INTEGRAL EQUATIONS (IE) FOR TRANSMISSION PROBLEMS IN LINEAR ELASTICITY AND BOUNDARY VALUE PROBLEMS OF CLASSICAL ELASTOSTATIC

In this kind of research, reducing transmission problems to boundary integral equations (IE) is a popular approach (Kupradze et al., 1979). A number of contexts benefit from the acknowledged substantial ellipticity of boundary integral equations: It proves the existence and convergence of approximations. This extreme ellipticity is examined in connection to boundary value issues by Costabel (1987), Costabel & Wendland (1986), Nédélec (1977), and a number of works like Wendland (1985). For any broad highly elliptic boundary value issue, including those with non-smooth (Lipschitz) borders and second order systems, it is feasible to generate strongly elliptic boundary integral equations. Costabel and Stephan investigated the Helmholtz equation in their 1985 paper on transmission problems and demonstrated that the boundary integral equations (IE) are significantly elliptic on smooth and flat polygonal edges. Costabel and Stephan (1988) found a basic concept in their research of electromagnetic wave scattering, demonstrating that the "direct method" results in a very elliptic set of integral equations (IE) when used to gearbox difficulties. For every particular set of boundary and transmission conditions, the idea posits that the consequent boundary integral equations (IE) are extremely elliptic. (Petersdorff, von, 1987)

We prove its validity in the general case of Lipschitz borders by applying it to the time-harmonic elastodynamic equations. Border potentials, whose densities are the jumps of the displacements and tractions over the transmission border T , are used to illustrate solutions of the transmission problem with locally limited energy. Our system of first-kind integral equations (IE) on the boundary manifold finds these densities as its solution. Integral operators with hypersingular, weakly singular, or strongly singular kernels are present in this system. A positive constant bilinear form valid down against

a compact perturbation is defined by this system, which is shown to meet a Garding inequality for the power norm. This is a feature of strongly elliptic curves.

One consequence of strong ellipticity is that the integral operator is Fredholm with zero index. Assuming there are no eigensolutions to the transmission issue, one may therefore prove existence. Conversely, according to Hildebrandt and Wienholtz (1964) and Stephan and Wendland (1976), all Galerkin approximation methods converge quasioptimally to the energy norm. Convergence holds in any finite dimensional subspace of the Sobolev spaces $JFf1/2(r)$ of the displacements, and of $H^{1/2}(T)$ of the tractions, as formulated in any of the standard boundary element methods, such as the h -, p -, and h - p versions of the boundary elements (Stephan and Suri, 1989, Wendland, 1985). The transmission issue may be constructively solved using our set of boundary integral equations.

For screen or crack scattering issues, one may use the same boundary integral operators (Wendland, 1985). For scattering from an inhomogeneous body, one can use the same operators (Costabel & Stephan, 1988, Costabel, 1987), which leads to a linked finite element/boundary element technique.

2.1 ENDURING INTEGRAL EQUATIONS (IE) AT THE BOUNDARY

The scattering of time-harmonic elastic waves at a homogeneous, isotropic scatterer, supplied by a bounded domain, may be solved using a boundary integral equation technique, which we describe here Ω_1 with unbounded exterior $\Omega_2 = \mathbf{R}^3 \setminus \bar{\Omega}_1$. We assume that the boundary $\Gamma = \partial\Omega_1$ of Ω_1 Lipschitz continuous.

Here we discuss the transmission issue in steady state elastodynamics, which goes like this: given vector fields, identify vector fields u_0 and t_0 on the boundary Γ that fulfill the equations of linear elasticity,

$$P_j u_j - \rho_j \omega^2 u_j = 0 \quad \text{in } \Omega_j, \quad j = 1, 2, \quad (2.1)$$

Together with the parameters of transmission

$$u_1 = u_2 + u_0, \quad t_1 = t_2 + t_0 \quad \text{on } \Gamma. \quad (2.2)$$

This is where we get the differential operators: P_j

$$P_j u \equiv -(\mu_j \Delta u + (\lambda_j + \mu_j) \text{grad div } u). \quad (2.3)$$

$\rho_j > 0$ is the density of the medium Ω_j , and $\omega > 0$ is the wave's frequency upon which it is incident. u_1, u_2 and u_0 signify the change in direction of the incident, dispersed, and refracted waves, correspondingly. All of the related tractions t_1, t_2, t_0 are given by $t_i = T_i(u_i)|_\Gamma$, where,

$$T_j(u) = 2\mu_j \partial_n u + \lambda_j n \text{ div } u + \mu_j n \times \text{curl } u$$

With Lamé constants $\mu_j > 0, 3\lambda_j + 2\mu_j > 0$. Here $\partial_n u$ is the derivative with respect to the outer normal n on T .

Along with (2.1) and (2.2), the displacement vector also requires a regularity requirement at infinity.

u_2 that we equalize by adding together a vector that is irrotational (lamellar) u_2^L and a solenoidal vector u_2^T . For $u_2 = u_2^L$ and $u_2 = u_2^T$, respectively, we require

If $u_2^* = u_2^L$ then we take for k_2^* the longitudinal (dilatational) wave number $k_2^L = \omega \rho_2^{1/2} (\lambda_2 + 2\mu_2)^{-1/2}$, whereas, if $u_2^* = u_2^T$ we take the transverse (shear) wave number $k_2^T = \omega \rho_2^{1/2} \mu_2^{-1/2}$.

We are interested in solutions u_j of (2.1), (2.2) which belong to $H_{\text{loc}}^1(\Omega_j)$, i.e., in solutions which have local finite energy. We define

$$\begin{aligned} \mathcal{L}_1 &= \{u_1 \in H^1(\Omega_1) : P_1 u_1 = \rho_1 \omega^2 u_1 \text{ in } \Omega_1\} \\ \mathcal{L}_2 &= \{u_2 \in H_{\text{loc}}^1(\bar{\Omega}_2) : P_2 u_2 = \rho_2 \omega^2 u_2 \text{ in } \Omega_2 \text{ and } u_2 \text{ satisfies} \end{aligned} \quad (2.4)$$

Given the trace lemma, it is worth mentioning that $u_j|_\Gamma \in H^{1/2}(\Gamma)$ for $u_j \in \mathcal{L}_j$. Also, we see that the tractions that are closely related t_i are defined in the usual way as distributions in the usual way as distributions in $H^{-1/2}(\Gamma)$ via the First Green formula (Costabel, 1988, Grisvard, 1985)

LEMMA 2.1. Let $u \in H_{\text{loc}}^1(\bar{\Omega}_j)$ with support compact $P_j u \in L_{\text{loc}}^2(\bar{\Omega}_j)$ and let $v \in H^1(\Omega_j)$ satisfy with bounded support. Then $T_j u|_\Gamma \in H^{-1/2}(\Gamma)$ is defined by

$$\int_{\Omega_j} P_j u \cdot v \, dx = (-1)^j \langle T_j u, v \rangle + \Phi_j(u, v) \quad (2.5)$$

with

$$\Phi_j(u, v) = \int_{\Omega_j} \sum_{i,h,k,l=1}^3 a_{ihkl}^j \varepsilon_{kl}(u) \varepsilon_{ih}(v) \, dx.$$

Here we have

$$a_{ihkl}^j = \lambda_j \delta_{ih} \delta_{kl} + \mu_j (\delta_{ik} \delta_{hl} + \delta_{il} \delta_{hk})$$

with $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$.

The brackets $\langle \cdot, \cdot \rangle$ denote the duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ which gives

$$\langle f, g \rangle = \int_\Gamma f \cdot g \, ds$$

For smooth vector fields f, g . The bilinear form $\Phi_j(u, v)$ represents the local displacement work due to the strain tensor

$$\varepsilon_{kl}(u) = \frac{1}{2} (\partial_k u_l + \partial_l u_k).$$

Generally speaking, the following uniqueness conclusion holds for the transmission problems (2.1), (2.2). (Citing Kupradze, we are thinking about the "basic contact problem for steady elastic oscillations.")

LEMMA 2.2. The only solution $u_i \in \mathcal{L}_i, i = 1, 2$, of the homogeneous problem corresponding to (2.1)-(2.3) is the trivial solution $u_1 \equiv 0, u_2 \equiv 0$.

By applying the symmetry Φ_j of to (2.5), one may get the Second Green formula, which is valid since the coefficients a_{ihkl}^j possess symmetry.

$$\int_{\Omega_j} (P_j u \cdot v - u \cdot P_j v) dx = (-1)^j \int_{\Gamma} (v \cdot T_j(u) - u \cdot T_j(v)) ds. \quad (2.6)$$

This gives, in Ω_j , with the fundamental solution $G_j(x, y, \omega)$ of $(P_j - \rho_j \omega^2)u_j = 0$, the Somigliana representation formula for $x \in \Omega_j$:

$$u_j(x) = (-1)^j \int_{\Gamma} \{T_j(x, y, \omega)v_j(y) - G_j(x, y, \omega)\phi_j(y)\} ds(y), \quad (2.7)$$

It follows that (2.7) is true for a Lipschitz border from the study in (Costabel, 1988).

LEMMA 2.3. Let $u_i \in \mathcal{L}_i$. Then the representation formula (2.7) holds for u_i in Ω_i . For any $v_i \in H^{1/2}(\Gamma)$ and any $\phi_i \in H^{-1/2}(\Gamma)$ the formula (2.7) defines a vector field $u_i \in \mathcal{L}_i$.

One obtains the relations on using Cauchy data in equation (2.7)

Γ .

$$\begin{pmatrix} v_j \\ \phi_j \end{pmatrix} = \mathcal{C}_j \begin{pmatrix} v_j \\ \phi_j \end{pmatrix}, \quad (2.8)$$

Where the Calderón projector

$$\mathcal{C}_j = \begin{pmatrix} \frac{1}{2} + (-1)^j \Lambda_j & -(-1)^j V_j \\ -(-1)^j D_j & \frac{1}{2} - (-1)^j \Lambda'_j \end{pmatrix} \quad (2.9)$$

Via the operators that determine the boundary's integrals

$$\begin{aligned} V_j v(x) &= \int_{\Gamma} G_j(x, y, \omega) v(y) ds(y), \quad \Lambda_j v(x) = \int_{\Gamma} T_j(x, y, \omega) v(y) ds(y), \\ D_j v(x) &= -T_{j,x} \int_{\Gamma} T_j(x, y, \omega) v(y) ds(y), \\ \Lambda'_j v(x) &= \int_{\Gamma} T_j(y, x, \omega)^T v(y) ds(y). \end{aligned} \quad (2.10)$$

We remark that, due to (Costabel, 1988, Costabel, 1984), the Calderón projector $\mathcal{C}_j$ is welldefined for boundary data (v_j, ϕ_j) belonging to the space $\mathcal{H} = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

The mapping properties of the Calderón projector $\mathcal{C}_j$ are described as follows.

LEMMA 2.4. (a) I and (ii) are $(v, \psi) \in \mathcal{H}$ mutually exclusive statements:

- (i) (v, ψ) are Cauchy data of some $u_j \in \mathcal{L}_j$;
- (ii) $(I - \mathcal{C}_j) \begin{pmatrix} v \\ \psi \end{pmatrix} = 0$.

(b) The operators that are used to map $\mathcal{H}$ onto its subspace of Cauchy data of weak solutions in take projection operators into account $\mathcal{L}_j$.

Because of Lemma 2.3 and the Calderón projector specification, the proof follows immediately from the representation formula (2.7) $\mathcal{C}_j$.

The transmission problems (2.1), (2.2) may so be expressed in the same way.

$$(I - \mathcal{C}_1) \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} = 0 \tag{2.11}$$

$$(I - \mathcal{C}_2) \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} = 0 \tag{2.12}$$

$$\begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} - \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \tag{2.13}$$

For these four vectors, we have a system of six equations. Our technology allows us to extract a square subsystem using $\begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}$ the formula (2.13), which is obtained by substituting it into (2.12) and then removing (2.11) from the final equation.

The integral equation (IE) governing the boundary

$$A \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} = (I - \mathcal{C}_2) \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \quad \text{with} \quad A := \mathcal{C}_1 - \mathcal{C}_2. \tag{2.14}$$

Here is an equivalency theorem that we have.

THEOREM 2.5. Let $\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \in \mathcal{H} = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ be given. Then there holds

(i) If $u_j \in \mathcal{L}_j$ finally resolve the transmission issue (2.1)-(2.3),

$$\begin{pmatrix} v \\ \phi \end{pmatrix} := \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} u_1|_{\Gamma} \\ T(u_1|)|_{\Gamma} \end{pmatrix} \in \mathcal{H}$$

solves the boundary integral equation (2.14).

(ii) If $\begin{pmatrix} v \\ \phi \end{pmatrix} \in \mathcal{H}$ solves (2.14), then, with

$$\begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} := C_1 \begin{pmatrix} v \\ \phi \end{pmatrix} \text{ and } \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} := C_2 \left(\begin{pmatrix} v \\ \phi \end{pmatrix} - \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \right)$$

and u_j defined by (2.7), $u_j \in \mathcal{L}_j$ solve the problem (2.1)–(2.3).

And u_j defined by (2.7), $u_j \in \mathcal{L}_j$ solve the problem (2.1)-(2.3).

Proof, (i) is derived from the previous proof of (2.14).

(ii). From the definition of $\begin{pmatrix} v_j \\ \phi_j \end{pmatrix}$ as well as the following projection C_j property:

$$(I - C_j) \begin{pmatrix} v_j \\ \phi_j \end{pmatrix} = 0,$$

Hence $\begin{pmatrix} v_j \\ \phi_j \end{pmatrix}$ are Cauchy data of certain $u_j \in \mathcal{L}_j$ which are then given by the representation formula (2.7).

In order to proceed, we must prove that the transmission requirement (2.2) is met:

$$\begin{aligned} \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} &= (C_2 - C_1) \begin{pmatrix} v \\ \phi \end{pmatrix} - C_2 \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} = -A \begin{pmatrix} v \\ \phi \end{pmatrix} - C_2 \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \\ &= -(I - C_2) \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} - C_2 \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} = - \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix}. \quad \square \end{aligned}$$

Remark. If $\begin{pmatrix} v \\ \phi \end{pmatrix}$ is an answer to the boundary integral equation (2.14), then the representation formula (2.7) yields the solution U_j to the transmission issue $\begin{pmatrix} v \\ \phi \end{pmatrix}$ meaning "source term," however, it is not always the case that $\begin{pmatrix} v \\ \phi \end{pmatrix} = \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}$ holds. Therefore, the singularity of the transmission problem has no direct impact on the boundary integral equations (Lemma 2.2). However, keep in mind the contrast

$$\begin{pmatrix} w \\ \psi \end{pmatrix} := \begin{pmatrix} v \\ \phi \end{pmatrix} - \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}.$$

There holds

$$\begin{pmatrix} w \\ \psi \end{pmatrix} = (I - C_1) \begin{pmatrix} v \\ \phi \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} w \\ \psi \end{pmatrix} &= (I - C_2 - A) \begin{pmatrix} v \\ \phi \end{pmatrix} = (I - C_2) \begin{pmatrix} v \\ \phi \end{pmatrix} - (I - C_2) \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \\ &= (I - C_2) \left(\begin{pmatrix} v \\ \phi \end{pmatrix} - \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \right). \end{aligned}$$

Therefore $C_1 \begin{pmatrix} w \\ \psi \end{pmatrix} = 0 = C_2 \begin{pmatrix} w \\ \psi \end{pmatrix}$, and this shows that $\begin{pmatrix} w \\ \psi \end{pmatrix}$ are Cauchy data of a homogeneous transmission problem for which the roles of P_1 and P_2 are interchanged:

$$\begin{pmatrix} w \\ \psi \end{pmatrix} = \begin{pmatrix} w_1 \\ T_2(w_1) \end{pmatrix} = \begin{pmatrix} w_2 \\ T_1(w_2) \end{pmatrix} \quad \text{on } \Gamma,$$

where $w_j \in \mathcal{L}_j$ satisfy

$$P_2 w_1 = \rho_2 \omega^2 w_1 \quad \text{on } \Omega_1; \quad P_1 w_2 = \rho_1 \omega^2 w_2 \quad \text{in } \Omega_2.$$

From Lemma 2.2 we infer that $w_1 = w_2 = 0$, hence $\begin{pmatrix} w \\ \psi \end{pmatrix} = 0$. Finally this implies $\begin{pmatrix} v \\ \phi \end{pmatrix} = \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}$ and, by extension, the singularity of the integral equation's solution (2.14). So, in order to prove that the second equation is unique, we must prove that the first transmission issue and the aforementioned "adjoint" transmission problem are both unique.

Theorem 2.6. A highly elliptic operator A exists, $\gamma > 0$ and there is a compact operator $T : \mathcal{H} \rightarrow \mathcal{H}$

$$\operatorname{Re} \left\langle (A + T) \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle \geq \gamma \left(\|v\|_{H^{1/2}(\Gamma)}^2 + \|\phi\|_{H^{-1/2}(\Gamma)}^2 \right) \quad (2.15)$$

for all $\begin{pmatrix} v \\ \phi \end{pmatrix} \in \mathcal{H}$. Here the brackets denote the natural (anti-) duality of $\mathcal{H}$ with itself:

$$\left\langle \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix} \right\rangle := \int_{\Gamma} (\bar{v}\psi + w\bar{\phi}) \, ds \quad \text{for } \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix} \in \mathcal{H}.$$

PROOF. We write

$$A = A_1 + A_2 \quad \text{with} \quad A_j = (-1)^j \left(\frac{1}{2} - C_j \right) = \begin{pmatrix} -\Lambda_j & V_j \\ D_j & \Lambda_j' \end{pmatrix}.$$

Proving that the operators are highly elliptic is sufficient, as the result is a sum of two operators that are themselves strongly elliptic (v, ϕ) . Let then u_j , $j = 1, 2$, be defined by

$$u_j(x) = \chi(x) \int_{\Gamma} \{T_1(x, y)v(y) - G_1(x, y)\phi(y)\} ds(y), \quad x \in \Omega_j.$$

Here we choose $\chi \in C_0^\infty(\mathbf{R}^3)$ satisfying $\chi \equiv 1$ in a neighborhood of $\bar{\Omega}_1$. Then, by definition of the Calderón projectors (i.e., the classical jump relations for the elastic potentials), the Cauchy data

$$v_j := u_j|_{\Gamma} \quad \text{and} \quad \phi_j := T(u_j)$$

Satisfy

$$\begin{pmatrix} v_j \\ \phi_j \end{pmatrix} = (-1)^j \left(\frac{1}{2} - (-1)^j A_1 \right) \begin{pmatrix} v \\ \phi \end{pmatrix}.$$

By adding and subtracting these two equations, we find

$$\begin{aligned} A_1 \begin{pmatrix} v \\ \phi \end{pmatrix} &= - \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} - \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}, \\ \begin{pmatrix} v \\ \phi \end{pmatrix} &= - \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} + \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}. \end{aligned}$$

Thus the bilinear form defined by A_1 is given by

$$\begin{aligned} & 2 \left\langle A_1 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} + \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} - \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} \right\rangle \\ &\quad + \left\langle \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} \right\rangle \\ &= 2 \operatorname{Re} \int_{\Gamma} (\bar{v}_1 \phi_1 - \bar{v}_2 \phi_2) ds + 2i \operatorname{Im} \int_{\Gamma} (\bar{v}_2 \phi_1 - \bar{v}_1 \phi_2) ds. \end{aligned}$$

Hence

$$\operatorname{Re} \left\langle A_1 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle = \operatorname{Re} \int_{\Gamma} (\bar{v}_1 \phi_1 - \bar{v}_2 \phi_2) ds. \quad (2.16)$$

Now we need the first Green formulas for P_1 in Ω_1 and Ω_2 . This leads to

$$\tilde{\Phi}_j(u_j, u_j) - \int_{\Omega_j} \bar{u}_j \cdot (P_1 - \rho_1 \omega^2) u_j \, dx = -(-1)^j \int_{\Gamma} \bar{v}_j \phi_j \, ds, \quad (2.17)$$

Where

$$\tilde{\Phi}_j(u_j, u_j) = \int_{\Omega_j} (\Sigma a_{ihkl}^1 \overline{\varepsilon_{kl}(u_j)} \varepsilon_{ih}(u_j) - \rho_1^2 \omega |u_j|^2) \, dx.$$

Now $P_1 u_1 - \rho_1^2 \omega u_1 = 0$ and $P_1 u_2 - \rho_1^2 \omega u_2 = f_2$, where $f_2 \in C_0^\infty(\Omega_2)$. ($f_2 \equiv 0$ whenever $\chi \equiv 1$ or $\chi \equiv 0$ holds.) From (2.16) and (2.17) together we find

$$(2.18) \quad \operatorname{Re} \left\langle A_1 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle = \operatorname{Re} \{ \tilde{\Phi}_1(u_1, u_1) + \tilde{\Phi}_2(u_2, u_2) - \int_{\Omega_2} \bar{u}_2 \cdot f_2 \, dx \}.$$

As the support of f_2 is disjoint from Γ , there is a compact operator T_1 on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\left| \int_{\Omega_2} \bar{u}_2 \cdot f_2 \, dx \right| \leq \left\langle T_1 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle.$$

From Korn's inequality and the trace lemma we find that there exist compact quadratic forms k_j on $H^1(\Omega_j)$ and hence a compact operator T_2 on $\mathcal{H} = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \tilde{\Phi}_1(u_1, u_1) + \tilde{\Phi}_2(u_2, u_2) &\geq \gamma_1 \left(\|u_1\|_{H^1(\Omega_1)}^2 + \|u_2\|_{H^1(\Omega_2)}^2 \right) \\ &\quad - k_1(u_1) - k_2(u_2) \\ &\geq \gamma_2 \left(\|v\|_{H^{1/2}(\Gamma)}^2 + \|\phi\|_{H^{-1/2}(\Gamma)}^2 \right) \\ &\quad - \left\langle T_2 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle. \end{aligned}$$

From (2.18) and (2.19) together we get the desired result:

$$\operatorname{Re} \left\langle (A_1 + T_1 + T_2) \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle \geq \gamma_2 \left(\|v\|_{H^{1/2}(\Gamma)}^2 + \|\phi\|_{H^{-1/2}(\Gamma)}^2 \right).$$

With respect to Garding's inequality (2.15), operator A maps $\mathcal{H}$ into itself and has an index of zero. The uniqueness of the integral equation (IE) (2.14) proves that there must

be a solution to the transmission problem (Theorem 2.5). On the other side, the assumptions in Lemma 2.2 suggest that there is a one-of-a-kind solution to the gearbox problem. As stated in Theorem 2.5, the solutions to the integral equation (IE) (2.14), which are equal to the transmission problems (2.1), (2.2), and the integral equation, do not change.

2.2 INTEGRAL EQUATION (IE) APPROACH TO CLASSICAL ELASTOSTATIC BOUNDARY VALUE PROBLEMS

Famous Italian school geometers like used the similarities between potential theory and classical elasticity (Betti, 1872, Somigliana, 1885, Lauricella, 1909). One may argue that Green's and Poisson's potential theory methods evolved into Betti's more generic strategy for solving the elasticity equations. Both Fredholm (1905) and Lauricella (1909) made significant contributions to the powerful! address problems with elasticity using theory of singular integral equations. An intrinsic difficulty of Betti's method is to construct the tensor version of a Green's function; this alternative approach somewhat sidesteps this issue. In subsequent years, planar elasticity issues have been addressed using the intricate method for solving single integral equations (IE) (Muskhelishvili, 1953; Sherman, 1940).

On the other hand, two-dimensional elasticity issues may be effectively tackled utilising real variable techniques that employ integral equations. The statement is based on Jaswon's paper (Jaswon, 1963). Although it is a tough task even for connected domains, his method does not need finding effect functions and mapping functions for the specified sections. Nevertheless, it employs a stress function ϕ rather than the explicitly given boundary values and is inherently two-dimensional. Thankfully, our research shows that it is possible to formulate problems utilising the given boundary values in a real variable form.

To effectively comprehend the suggested approach, one must find out the basic solution of the Navier-Cauchy equilibrium equations in n -dimensional space, where n is the dimensionality of space, and t is the dimension of the concentrated force applied. The result of this solution is a Laplace equation identity, parallel to the third of Green.

The representation of the boundary needs the knowledge of a field point on the boundary contour, the associated traction and displacement of the boundary. One of these quantities of the boundary is required to be given in more detail than the other, in principle. The novelty of this study is that it shows that the formulation naturally couples these two variables of the boundary and the result is a coupled system of integral equations (IE) that can be solved in tandem when suitable conditions of the boundaries are given. These issues may be categorized as first-kind, second-kind and mixed boundary value problems depending on the type of the prescribed data. The differential equations contain implicit variables; the boundary tractions or displacements. The first stage in constructing the displacement field is to get the vector identities. Jaswon and Symm (1963) note that the numerical solutions to similar but simpler integral equations (IE) that arise in Green as a result of the third identity on the border (Green boundary formula) have been calculated.

For highly connected locations, there is no fundamental restriction on the types of issues that may be analysed. No separate traction balancing on each border or search for a mapping function are required. The formulation also starts by removing solutions that include Volterra dislocations. Consequently, the three basic boundary value issues of elasticity are addressed in a consistent and transparent way. A single formulation based just on boundary data may accomplish the generalisability, practicality, ease, and economy of effort for any problem.

This study uses numerical data from a subset of situations and focuses only on two-dimensional challenges to strengthen the method's trustworthiness. Using data from a wide variety of topics, the technique will be evaluated and explored in the next papers. The boundary formula approach has the remarkable conceptual advantage of being applicable to three dimensions in addition to two, as integral equations (IE) with identical specifications evolve in both cases. In addition to alternative convertible auxiliary function formulations, including one derived from the Papkovitch-Neuber decomposition, academics are now investigating the technique's three-dimensional counterpart.

2.2.1 Formulation of integral equations

When considering elasticity without considering body forces, the Navier-Cauchy equations are expressed as

$$(\lambda + \mu)\vartheta_{,i} + \mu\nabla^2 u_i = 0, \quad \vartheta = u_{i,i} \quad (2.18)$$

assuming that all variables are functions of the Cartesian plane in x -coordinates, and assuming that generalised strain and stress on the plane are problematic, Equation (2.1) employs the standard vector notation, with u standing for displacement and ϑ for dilatation.

Let the body under consideration first be finite and bounded by a single smooth contour C which, as described by (Muskhelishvili, 1953), admits a representation in the form $x_i = x_i(s)$. The first derivatives $x_{i,s}(s)$ are continuous and never simultaneously zero. The parameter s is arc length along the contour from some arbitrary origin on it. The region occupied by the body is traversed in the positive sense keeping the region on the left. The positive tangent at a boundary point Q agrees in sense with that of positive traverse and the positive unit normal n_i at Q points away from the region. The interior open region of the plane thus defined is labelled σ' and the exterior open region σ'' . Points of σ' are designated p , points of σ'' are designated p' . Points of C are designated P or Q .

For the body ($\sigma + C$) while subjected to boundary tractions t and maintaining equilibrium, the relation

$$\Gamma[u_i] = t_i = \lambda\vartheta n_i + \mu n_j(u_{i,j} + u_{j,i}) \quad (2.19)$$

Must remain attached to C , where r is the operator given.

Assume that x and t are two locations σ' on and denote

$$r = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2]^{1/2}.$$

Further, let U_{ij} be the tensor field defined by

$$U_{ij} = \delta_{ij} \log r + Mr_{,i}r_{,j}.$$

The displacement vectors u'_i defined by

$$u'_i = U_{ij}e_j$$

in which e_i are unit base vectors both satisfy Eqs. (2.1) in σ provided

$$r \neq 0, \quad M = -\frac{(\lambda + \mu)}{(\lambda + 3\mu)}.$$

These displacement vectors arise from concentrated forces of magnitude $-4\pi\mu(\lambda+2\mu)/(\lambda+3\mu)$ acting at the point $p = \xi_i$ in the x_i directions respectively. These singularities are termed nuclei of strain by Love [11].

The traction vectors t'_i on C corresponding to u'_i are computed from (2.2) and may be written

$$t'_i = T_{ij}e_j$$

in which

$$T_{ij} = \frac{\partial}{\partial n_Q} \log r [k\delta_{ij} - 4\mu Mr_{,i}r_{,j}] + k[(\log r)_{,i}n_j - (\log r)_{,j}n_i].$$

In the above expression for the tensor field T_{ij} , With regard to the coordinates x , at Q , and $r = r(p, Q)$, the derivatives are computed. On top of that, k is worth $2\mu/(\lambda + 3\mu)$.

The reciprocal work theorem of Betti, in its vectorial version, may be expressed as

$$\int_{C+m} (u_i T_{ij} - t_i U_{ij}) dQ = 0 \quad (2.20)$$

Assuming U and T are distinct, not included in a small circle m with a radius of δ is the point $p = t$ of ∂T , where dQ is an element of arc length at Q and $r = 0$. This theorem is predicated on the assumption that the vectors and their derivatives are sufficiently continuous and nonsingular. The generalisability is maintained by considering the vectors u and t to be zero, and by describing a balanced stress condition.. Everyone should now be able to see that

$$\lim_{\delta \rightarrow 0} \int_m t_i U_{ij} dQ = 0, \quad \lim_{\delta \rightarrow 0} \int_m u_i T_{ij} dQ = -\alpha^{-1} u_i(p) \quad (2.21)$$

in which $\alpha = (\lambda + 3\mu)/4\pi\mu(\lambda + 2\mu)$. Equation (2.3) yields, therefore, the expression

$$u_i(p) = \alpha \int_C [u_i(Q)T_{ij}(p, Q) - t_i(Q)U_{ij}(p, Q)] dQ \quad (2.22)$$

This is the integral of Somigliana's equivalent on the aircraft.

When applying Betti's theorem to cases where the occupied body area is $(cl + C)$, caution is required. Equation (2.22) is used for p' instead of p according to Betti's theorem, such that $u_i(p') = o(1)$ as $p' \rightarrow \infty$, and the tractions are balanced on C , 2.

Interdependent in equation (2.22) are the values of the variables u and (p) that satisfy equation (2.18), together with the boundary tractions t and (Q) that accompany them. Similarly to how Green's third identity of potential theory enables one to represent a harmonic function using the normal derivative and values at its border, this connection facilitates the same.

The integrals present in Eq. (2.22), i.e.,

$$\phi_i(p) = \int_C t_i(Q)U_{ij}(p, Q) dQ, \quad \psi_i(p) = \int_C u_i(Q)T_{ij}(p, Q) dQ$$

Like the scalar integrals of the single and double layers in potential theory, they seem similar. When the boundary C is traversed, the necessary integral characteristics are satisfied.

$$\lim_{p \rightarrow P} \phi_i(p) = \phi_i(P), \quad \lim_{p \rightarrow P} \psi_i(p) = \psi_i(P)$$

then letting $p \rightarrow P$ in the second of these integrals yields

$$\psi_i(P) = \pi(k - 2\mu M)u_i(P) + \int_C u_i(Q)T_{ij}(P, Q) dQ.$$

Assuming u satisfies a Holder condition on C and knowing the singular integral as a Cauchy Principal Value makes verifying this conclusion straightforward. For the integral of the one-layer type, a simpler and similar limiting approach is available.

$$\phi_i(P) = \int_C t_i(Q)U_{ij}(P, Q) dQ.$$

Since

$$u_i(P) = \alpha[\psi_i(P) - \phi_i(P)]$$

and since

$$\alpha\pi(k - 2\mu M) = 1/2$$

there results

$$K[u_i] = u_i(P) + 2\alpha \int_C [t_i(Q)U_{ij}(P, Q) - u_i(Q)T_{ij}(P, Q)] dQ = 0 \quad (2.23)$$

While using the definition of Poisson's ratio given by Muskhelishvili (1953) to singular operators, the operator K is regular for all admissible values. Equivalent to Green's boundary formula in elasticity theory is (2.23).

An integral equation (IE) for the variable u on C given on C (traction problem) is provided by the two halves of Eq. (2.23). As expected, the stiff body's displacement vector

$$u_i(P) = a_i + b e_{i3} x_3(P)$$

Meets the homogeneous version of these equations, that is, since

$$u_i(P) - 2\alpha \int_C u_i(Q) T_{ij}(P, Q) dQ = 0 \quad (2.24)$$

where a_i and b are arbitrary constants. Application of the Fredholm alternative yields

$$\int_C t_i(Q) dQ = \int_C e_{3jk} x_j(Q) t_k(Q) dQ = 0 \quad (2.25)$$

Provide the prerequisites for Eq. (2.23)'s solubility. This is, without a doubt, the moment when the human body finds balance.

Although Eqs. (2.23) were used in the preceding section, it should be remembered that these are

Not Fredholm integral equations. The phrase's existence

$$T_{ij}^* = k[(\log r)_{,i} n_j - (\log r)_{,j} n_i]$$

in the kernel function T_{ij} is responsible for this. More precisely

$$\lim_{Q \rightarrow P} |Q - P| T_{ij}^*(P, Q) \neq 0$$

Conversely, zero is required for Fredholm kernels. The index K (see to Muskhelishvili, 1953, Chapter HJ) computed from T ; stays zero, and the Fredholm alternative is appropriate regardless.

Eliminating the stiff body part is a numerical need for fixing the traction problem. Keep reading the section about numerical processes to find out more about this method. The two-part integral equation (IE) in the C-based variables t is given by (2.23), where u is a variable and the displacement issue is at hand. There ought to be one unique solution f ; on C to these equations for each u ; on C . The homogeneous equations must be, therefore,

$$\int_C t_i(Q) U_{ij}(P, Q) dQ = 0$$

have no nonzero solutions. This may be proven as follows. Let

$$u_i^*(p) = \int_C \lambda_i(Q) U_{ij}(p, Q) dQ, \quad \int_C \lambda_i(Q) dQ = 0$$

be a displacement field defined in σ . The field $u_i^*(p')$ is also defined in σ' and vanishes uniformly as $p' \rightarrow \infty$. Further, let

$$\lim_{p \rightarrow P} t_i^*(p) = t_i^*(P)^+, \quad \lim_{p' \rightarrow P} t_i^*(p') = t_i^*(P)^-$$

in which

$$\begin{aligned} t_i^*(P)^+ &= k_1 \lambda_i(P) + \int_C \Gamma[\lambda_i(Q) U_{ij}(P, Q)] dQ, \\ t_i^*(P)^- &= -k_1 \lambda_i(P) + \int_C \Gamma[\lambda_i(Q) U_{ij}(P, Q)] dQ \end{aligned} \quad (2.26)$$

As described by Eq. (2.19), with r being the vector operator and k_1 being a nonzero constant. If the field u is zero everywhere in the plane, then u is continuous across C and $u(P) = 0$. Thus

$$t(P)^+ - t(P)^- = 0$$

By deducting the second equation from the first, we get

$$X_i(P) = 0.$$

The evidence is now complete.

We apply to one of the pieces of C the knowledge that $u_i = u_i$ and to the other piece of C the knowledge that $t_i = t_i$: so as to form a system of four equations of the unknowns u_i and t_i on C , which enables us to solve the mixed problem. In order to construct the displacement field for this and the two preceding boundary value problems, we use Equation (2.22).

By applying what we know about the variables u_i and t_i to a system of four equations

on C , we may solve the mixed problem. On one piece of C , u is equal to u , and on the other, t_1 is equal to t . Equation (2.23) is used to build the displacement field for this and the two previous boundary value issues.

2.2.2 Numerical techniques

This study updates and extends methods for estimating the integral equations (IE) given in (Symm, 1963). When comparing Symm's kernels to the more complex ones that emerge in Eqs. (2.23), it is important to keep in mind that their integrals only exist in the Principal Value sense.

Assume for the sake of argument that the displacement issue needs resolution. In contour C , there is a set of n nodes, and each interval on the contour is of length n . The center of the μ th interval might be where you find Q , the μ th node point. Presuming the unknown tractions $t_i(Q)$ are constant allows for integration over all intervals. To substitute it, we may use the linear algebraic system and look at equations (2.23).

$$\sum_{\mu=1}^n t_{i\mu}(Q_\mu) \int_{\mu} U_{ij}(P, Q) dQ = \Theta_j(P) \quad (j = 1, 2, \dots, n) \quad (2.27)$$

in which $\Theta_j(P)$ are known functions involving the assigned boundary displacements. The end points of the indicated μ th interval of integration lie midway between $Q_{\mu-1}$ and $Q_{\mu+1}$ and these end points are denoted by $Q_{\mu-1/2}$ and $Q_{\mu+1/2}$ respectively.

The solutions $t_{i\mu}$ of the system (3.1) give rise to a displacement field

$$u_i(p) = -\alpha \left[\sum_{\mu=1}^n t_{i\mu}(Q_\mu) \int_{\mu} U_{ij}(p, Q) dQ - \Theta'_j(p) \right] \quad (2.28)$$

in which

$$\Theta'_j(p) = \int_C u_i(Q) T_{ij}(p, Q) dQ$$

is known.

To evaluate the coefficients of l_{ij} in Eqs. (2.27) note that point P_i will coincide with one of the points Q_j (2.27) just once in all system equations. The integrals in (2.27) are well-defined for the remaining possibilities, thus a Simpson approximation is utilized. During when

$P_i = Q_j$, i.e., P_i encompasses the integration interval, the concept

$$\int_{\mu} \log r \, dQ = r(Q_{\mu}, Q_{\mu-1/2})[\log r(Q_{\mu}, Q_{\mu-1/2}) - 1] \\ + r(Q_{\mu}, Q_{\mu+1/2})[\log r(Q_{\mu}, Q_{\mu+1/2}) - 1]$$

is found by direct integration assuming $ds \doteq dr$ for the appropriate term in Eqs. (2.27). For the remaining integrals in Eqs. (2.27) it is helpful to note that

$$r_{,1} = \cos \theta, \quad r_{,2} = \sin \theta$$

where θ is the angle between $r(P, Q)$ and the x_1 axis. Although θ is not well defined when $r = 0$, the integrals

$$\int_{\mu} r_{,i} r_{,j} \, dQ$$

do exist. But because of this character for $r = 0$, a trapezoidal approximation over the interval $Q_{\mu-1/2}, Q_{\mu+1/2}$ is used for such integrals when $P_{\mu} = Q_{\mu}$.

To solve the traction problem Eqs. (2.23) are replaced by the system

$$u_i(P_{\mu}) - 2\alpha \sum_{\mu=1}^n u_{i\mu}(Q_{\mu}) \int_{\mu} T_{ij}(P_{\mu}, Q) \, dQ = \Omega_i(P_{\mu}) \quad (2.29)$$

in which $\Omega_i(P_{\mu})$ are known functions involving the assigned boundary tractions. To ensure a unique solution to the system (2.29) it is necessary to specify the rigid body-motion. This can be accomplished by suitably prescribing three displacements components of three convenient boundary points as is specifically demonstrated in the next section. Three equations are then eliminated. The solutions $u_{i\mu}$ of this reduced system give rise to a displacement field

$$u_i(p) = \alpha \left[\sum_{\mu=1}^n u_{i\mu} \int_{\mu} T_{ij}(p, Q) \, dQ - \Omega'_i(p) \right] \quad (2.30)$$

in which

$$\Omega'_i(p) = \int_c t_i(Q) U_{ij}(p, Q) \, dQ$$

is known.

Regarding the coefficients of $u_{i\mu}$ in Eqs. (2.29) note that since $(\partial/\partial n_Q) \log r \, dQ = d\theta$, and since $r_{,i}$ have the interpretations $\cos \theta, \sin \theta$, integrations may be carried out exactly for P_{μ} outside the interval of integration. For P_{μ} within the interval of integration, the integrals

$$\beta_{\mu ij} = \int_{\mu} T_{ij}^0(Q_{\mu}, Q) \, dQ,$$

in which $T_{ij}^0 = T_{ij} - T_{ij}^*$, are more conveniently evaluated by the equivalent calculation

$$\beta_{\mu ij} = \int_c T_{ij}^c(Q_{\mu}, Q) \, dQ - \int_{c-\mu} T_{ij}^0(Q_{\mu}, Q) \, dQ. \quad (2.31)$$

Using the known relations

$$\int_C \frac{\partial}{\partial n_Q} \log r \, dQ = \pi, \quad \int_C r_{,i} r_{,i} \frac{\partial}{\partial n_Q} \log r \, dQ = \frac{\pi}{2} \delta_{ii}$$

The first integral of (2.31) is easily obtainable if r is defined as $r(P, Q)$. It is possible to bypass the first integration and go directly to the second. The only remaining components that need repair at this time are

$$\beta_{\mu ii}^* = \int_{\mu} T_{ii}^*(Q_{\mu}, Q) \, dQ$$

When assessed in the required sense of the Principal Value, have a common value of zero.

We know that for the mixed boundary value problem, the displacements $U_i = u_i$ and the remaining tractions $t_i = t_i$ of Part C are known. Different intervals of n_1 and n_2 divide curves C_1 and C_2 , respectively. Equations (2.6) are then substituted by the system.

$$\begin{aligned} u_i(P_r) + 2\alpha \left[\sum_{\mu=1}^{n_1} t_{i\mu}^1(Q_{\mu}) \int_{\mu} U_{ii}(P_r, Q) \, dQ - \sum_{\mu=1}^{n_2} u_{i\mu}^2(Q_{\mu}) \int_{\mu} T_{ii}(P_r, Q) \, dQ \right] \\ = \Omega_i^1(P_r) - \Theta_i^2(P_r) \end{aligned} \quad (2.32)$$

“In which $\Omega_i^1(P_r)$ and $\Theta_i^2(P_r)$ which are known to be functions of the provided tractions on C_2 and displacements on C_1 , respectively. The displacement field $u_i(p)$ is generated using the values of u_i on C_2 and t_i on C_1 in Equation. The integrations are carried out in the same way as in Equations (3.27) and (3.30).”

A digital computer, with the right inputs, can easily calculate any numerical operation.

$$Q_{\mu} = x_{i\mu}, \quad Q_{\mu \pm 1/2} = x_{i\mu \pm 1/2}.$$

Either type in 0 and $O(P_r)$ by hand or give the computer the option to do it automatically. You may achieve the required displacement u_i by writing a single program that first finds and solves the relevant equations.

1. Numerical data

In order to verify that the approximations described in Section 3 are correct, we find numerical solutions to many basic boundary value issues. This allows for a comparison of computed data with analytical answers that can be readily acquired. The four geometric shapes circle, ring, rectangle, and ellipse are taken into account as domains. All domains are considered to be problems with planar strain except the rectangular one. In order to make programming easier and more generic, the machine assumes that the tractions and displacements are constant across the μ th interval and computes all functions 0 and O accordingly. On top of that, programming makes no use of the apparent symmetry. Each nodal point's related quantities are calculated separately.

In each domain, (x) is used, the centroid is used as the origin in the circular domain and the centres determined using major and minor axes in the elliptical and rectangular domains.

We begin by thinking about a traction problem in the domain of circles. On the clock, there are twelve places labeled K_{dal} on the body, and they are all numbered so that we can identify the hour position. Therefore, point three is on the x_1 axis. The boundary is subjected to a tensile torque $N = P_0 \sin^2 \theta$ with P_0 being a constant and θ being the angle between the radius of the circle and the positive x -axis. It is a good measure to reduce the stiff body motion to zero and the θ component of associated boundary displacement is capped. The first step in verifying the findings is to apply a tensile torque, denoted as $N = P_0 \cos^2 \theta$, and compare the components of the net boundary displacement, J_{Cnt} , with those derived from the stress, $N = P_0$, acting alone. Adding a node halfway between each of the initial twelve nodes gives us twenty-four nodes, which allows us to run the problem again. Following a clockwise tour, the remaining nodes are numbered sequentially, with node one staying in the same place. As anticipated, there is an increase in the precision of the computed data. Table I only contains data for the first quadrant since the predicted symmetry in the border data was attained.

The U_2 and u_1 components of displacement at the x_1 and x_2 nodal locations, respectively, are set to zero in order to characterize the rigid body motion that is relevant

to this and subsequent traction problems. The boundary data becomes too particular when the units at nodes represent the displacement components of all material points on the corresponding boundary intervals. Little inaccuracy is introduced since arc lengths are somewhat small (even for twelve segments) and the relative displacement between intervals is really rather large. By locking the inner boundary and clockwise rotating the outside border $cJ,0$, we may solve the displacement issue of the circular ring. Twelve nodes in the solid circle correspond to the nodes in the inner boundary. Twelve more nodes are added to the periphery of the clock every half an hour. At half-past twelve, node thirteen joins the sequence and continues the counting. There are twenty-four more that are numbered in a clockwise fashion. We compare the calculated inner and outer border tractions $t!$ and $t!$, which represent the boundary displacement, to the analytical predictions. To create a complicated problem, remember to hold the inner bound when you apply $t!$.

Table 2.1: Information about the boundaries of a circular domain with traction $N = PO \sin 2 c/\rangle$

“Radius $p = 104$ in., Poisson's ratio $\nu = .25$ Young's Modulus $E = 107$ p.s.i., $P0 = 1600$ p.s.i.

Boundary u, μ (in.), $n = 12$ ”

Numerical Analytical

Node	$U1\mu$	$u,.$	$U1\mu$	$U2\mu$
12	0	1.2690	0	1.3333
1	0.2531	0.8689	0.3333	0.8660
2	-0.0219	0.2360	0.0001	0.1667
3	-0.2910	0	-0.3333	0

Boundary $u;\mu$ (in.), $n = 24$

Numerical Analytical”

Node	$U1\mu$	$u2.$	$U1\mu$	$U2\mu$	
23	0	1.2984	0	1.3333	
24	0.2075	1.1769	0.2357	1.2016	
1	0.291fi	0.8660	0.33:3	0.8660	
2	0.2013	0.4960	0.2357	0.4714	
3	-0.0120	0.2015	0.0001	0.1667	
4	-0.2244	0.0477	-0.2357	0.0231	
5	-0.3123	0	-0.3333	0	

Forever set. Next, the values $t!$ and $u!$ on the inner and outer circles are determined.

The data for the boundaries of the first quadrant may be found in Table II.

The ellipse's perimeter is formed by twenty-four equal segments. All of the nodes are numbered in a clockwise traversal sequence, with the first node being located on the positive x_1 axis. The calculated values are worked with the help of a specific program which involves the formula of a circle and the coordinates of the nodal points which are evenly spaced. Using a constant normal tensile load. $N=P$ 0 along the boundary, the displacement of the boundary components are also obtained. Table III provides the data for the first quadrant's boundaries.

The perimeter of the rectangle is made up of twenty-eight segments; nine of them run along the length of the rectangle and five along its width. Because of this, the nodal points will never be located on corners where the contour lacks a clear tangent. All of the nodes in the ellipse follow the same numerical scheme, and the first node is back on the positive x_1 axis.

Table 2.2: Limit information for a spherical ring

“Radii: $p_1 = 10'$ in., $p_2 = 2 \times 10'$ in.; Poisson's ratio $\nu = .25$

Young's Modulus $E = 107$ p.s.i.

Displacement prob., $\phi_0 = 5 \times 10^{-6}$ radian

Tractions $t_{i\mu}$ (p.s.i.)”

Numerical Analytical”

Node	11 μ	12 μ	11 μ	12 μ
12	-543.2	0	-532.0	0
1	-470.4	271.6	-461.5	266.0
2	-271.6	470.4	-266.0	461.5
3	0	543.2	0	532.0
13	140.4	-37.6	128.7	-34.5
14	102.7	-102.7	94.3	-94.3
15	37.6	-140.4	34.5	-128.7
-				

Mixed prob., uniform shear 532 p.s.i. on C_2

Mixed prob., uniform shear 532 p.s.i. on C_2
 Tractions $t_{i\mu}$ (p.s.i.)

Node	Numerical		Analytical	
	$t_{1\mu}$	$t_{2\mu}$	$t_{1\mu}$	$t_{2\mu}$
12	-497.2	0	-532.0	0
1	-430.6	248.6	-461.5	266.0
2	-248.6	430.6	-266.0	461.5
3	0	497.2	0	532.0

Displacements $u_{i\mu}$ (in.)

	$u_{1\mu}$	$u_{2\mu}$	$u_{1\mu}$	$u_{2\mu}$
13	0.8841	-0.2369	0.9659	-0.2588
14	0.6472	-0.6472	0.7071	-0.7071
15	0.2369	-0.8841	0.2588	-0.9659

Table 2.3: Boundary data for ellipse, $P_0 = -10'$ p.s.i.

Semiaxes: $a = 200$ in., $b = 100$ in.; Poisson's ratio $\nu = .20$,

Young's Modulus $E = 108$ p.s.i.

Boundary $u_{,11}$ (in.)

Numerical Analytical

Node	U_{111}	U_{211}	U_{111}	U_{211}
19	0	0.3541	0	0.3600
20	0.1434	0.3484	0.1454	0.3526
21	0.2849	0.3232	0.2887	0.3298
22	0.4226	0.2824	0.4277	0.2896
23	0.5507	0.2203	0.5587	0.2271
24	0.6554	0.1280	0.6689	0.1332
1	0.7008	0	0.7200	0

This domain's traction problem is solved simply by applying uniform normal tensile traction t_1 to its two shorter sides. Everything else is floundering. The boundary aspect and discontinuity of the needed traction are completely new to us, even if this is a really simple issue.

Table IV contains the predicted boundary displacement components and data for the first quadrant.

To ensure the accuracy of the field values, we computed the displacement components at various sites throughout the circular domain with boundary traction $N = P_0 \sin^2 \theta$. The agreement with the field values calculated analytically is as good as, if not better than, what is achieved for the specified boundary quantities. Also, be sure to verify the stress components! Due to the two layers of approximation needed in obtaining the identical coordinates using a simple finite difference approach, the accuracy was rather low.

unexpected. Once the boundary values are known, it is sufficient to differentiate Eq. (2.5) directly to get an equation for the stresses, hence the finite difference approach is unnecessary. This removes one tier of approximation. Only the kernel functions are affected by these differentiations, because they are all relative to the point p in the field.

Table 2.4: Boundary data for rectangular domain, $t = 10'$ p.s.i.

“Dimensions: 360 in. X 200 in.;

Poisson's ratio $\nu = .20$, Young's Modulus $E = 108$ p.s.i.

Boundary u, v (in.)

Numerical Analytical”

Node	u, v	U_{211}	U_{111}	U_{211}
22	0	-0.2208	0	-0.2000
23	0.3991	-0.2213	0.4000	-0.2000
24	0.7982	-0.2227	0.8000	-0.2000
25	1.1973	-0.2249	1.2000	-0.2000
26	1.5964	-0.2250	1.6000	-0.2000
27	1.7909	-0.1495	1.8000	-0.1600
28	1.8048	-0.0873	1.8000	-0.0800
1	1.8048	0	1.8000	0

On the premise that known boundary values remain constant on the ranges of the borders, the integrals required may be calculated using a similar technique as in the preceding section 3, and stress equations composed of Eq. (3.2) can be quickly generated.

Our current focus is on delving further into the details for the direct stress computations and doing more comprehensive and demanding testing of the proposed method for identifying the pertinent boundary parameters, Concerns about efficacy persist.

Starting with a modest value of n and examining the generated data as n rises is a straightforward test for any given issue. The smooth boundary data and simple domain shape are undoubtedly contributing factors to the rather good results given for a small number of nodal points. The programs and data fit on the IBM 7094 computer core, thus n was selected for convenience. The first technique tested approximations for the three forms of boundary value concerns for the defined areas, not optimal programming, convergence, or error analysis. Until ill-conditioning produces a particular difficulty, no one knows how much higher n can be pushed. Furthermore, either more computer storage space or at least some focus on optimizing the software are needed for substantial increases in n . Presently, tape storage increases computer time; nonetheless, There was no issue that required over three minutes of computer time.

Nodes are evenly spaced all the way around the perimeter, with the exception of the ring. In light of the comments made by Symm in 1963, this seems like a good idea for optimally conditioning the equations. The question of how uneven intervals might be utilized to better manage boundary data that is changing quickly also needs to be addressed. Naturally, any detailed analysis of the few data offered on the aforementioned topics would provide no definitive results. But the solutions given do show the breadth of the approach clearly. Improving numerical procedures can help overcome tactical challenges encountered in more extensive tasks.

An effective dimensional reduction, where all approximations occur only on the border of the domain, is inherent in the boundary formula technique. However, for the technology to be practically applicable, a rather high-speed computer is necessary. Due of their widespread availability, integral equations (IE) are no longer limited to theoretical inquiries. It is believed that integral equation (IE) techniques may quickly provide solutions to difficult boundary value issues that cannot be solved using other approaches.

RESEARCH METHODOLOGY

3.1 POTENTIAL FUNCTION

Solving the Laplace equation inside certain boundary conditions yields solutions to potential theory questions. We are addressing mixed-boundary circumstances in our thesis.

We analyzed the effects of heat on isotropic and orthotropic bodies as part of our study. A two-dimensional body with four Griffith cracks and unlimited dimensions was studied. The faces of the cracks should be filled with flux, and the remaining section should be filled with temperature.

Suppose that the cracks are parallel to the x -axis, with the respective openings facing the y -axis. A "multi-connected body" describes this arrangement. Over some intervals, one may provide potential functions and over other intervals, one can specify their derivatives.

3.2 ISOTROPIC BODY

In the case of an isotropic body, the potential function (temperature field) is assumed to be in the form of no internal heat source or sinks in the medium. No radiations or convection is there. Cracks occupy the medium as $y = 0, b < |x| < c, d < |x| < e$, which means that cracks are equi-spaced around y -axis. All functions vanish as $x \rightarrow \infty$. It is very simple to calculate the conduction over two dimensional body.

$$\Delta^2 T = 0, \Delta^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (3.1)$$

Where T is temperature. The solution of (3.1) is assumed as

$$T(x, y) = \int_0^\infty \phi(\xi) e^{-\xi y} \cos(\xi x) d\xi \quad (3.2)$$

While $\phi(\xi)$ is unknown function. T through (3.2) satisfies (3.1). The mixed boundary conditions are,

$$T(x,0) = \begin{cases} P_1(x), 0 \leq |x| \leq b \\ P_2(x), c \leq |x| \leq d \\ 0, e \leq |x| < \infty \end{cases} \quad (3.3)$$

$$\frac{\partial T(x,0)}{\partial y} = Q(x), b < |x| < c, d < |x| < e \quad (3.4)$$

Along with the boundary requirements, we also assume the symmetry of the body's geometry. Therefore, the expressions (3.3) through (3.4) transform into

$$T(x,0) = \begin{cases} P_1(x), 0 \leq x \leq b \\ P_2(x), c \leq x \leq d \\ 0, e \leq x < \infty \end{cases} \quad (3.5)$$

$$\frac{\partial T(x,0)}{\partial y} = \begin{cases} Q_1(x), b < x < c \\ Q_2(x), d < x < e \end{cases} \quad (3.6)$$

This quintuple integral equation is obtained by substituting 3.2 into (3.3) and (3.4).

$$\int_0^\infty \phi(\xi) \cos(\xi x) d\xi = \begin{cases} P_1(x), 0 \leq x \leq b \\ P_2(x), c \leq x \leq d \\ 0, e \leq x < \infty \end{cases} \quad (3.7)$$

$$\int_0^\infty \xi \phi(\xi) \cos(\xi x) d\xi = \begin{cases} Q_1(x), b < x < c \\ Q_2(x), d < x < e \end{cases} \quad (3.8)$$

The solution of (3.7) -(3.8) is obtained by the method of (Kushwaha, 1975)

$$\phi(\xi) = \left\langle \int_b^c g(t) + \int_d^e h(t) \right\rangle \frac{\sin \xi t}{\xi} dt - \frac{2}{\pi} \left\langle \int_0^b P_1'(t) + \int_c^d P_2'(t) \right\rangle \frac{\sin \xi t}{\xi} dt \quad (3.9)$$

And use the integral

$$\int_0^\infty \frac{\sin(\xi x) \cos(\xi y)}{\xi} d\xi = \begin{cases} \pi/2, x > y \\ \pi/4, x = y \\ 0, x < y \end{cases}$$

Then (3.9) will satisfy (3.7) if

$$\int_b^c g(t) dt = 0, \int_d^e h(t) dt = P_2(d) \quad (3.10)$$

If we take the integral and the results from (Kushwaha, 1982) and replace (3.9) with (3.8), we get

$$\int_0^\infty \frac{\sin(\xi x) \sin(\xi y)}{\xi} d\xi = \frac{1}{2} \log \left| \frac{x-y}{x+y} \right|$$

$$g(t) = \frac{1}{\pi^2} \frac{1}{\delta(t)} \left[\left\langle \int_b^c Q_1(x) - \int_d^e Q_2(x) \right\rangle \frac{x\delta(x)}{(x^2 - t^2)} dx + C_1 t^2 + C_2 \right] \quad (3.11)$$

Where $b < t < c$.

$$h(t) = -\frac{1}{\pi\delta(t)} \left[\left\langle \int_b^c Q_1(x) - \int_d^e Q_2(x) \right\rangle \frac{x\delta(x)}{(x^2 - t^2)} dx + C_1 t^2 + C_2 \right] \quad (3.12)$$

Where $d < t < e$.

Two arbitrary constants, C_1 and C_2 , may be found using the formula in (3.10). Then it becomes a breeze calculate flux over $0 \leq |x| \leq b, c \leq |x| \leq d, e \leq |x| < \infty$ and temperature over $b < |x| < c, d < |x| < e$.

3.3 ORTHOTROPIC BODY

A second-order PDE whose constant coefficients depend on specific heat and other parameters governs the heat conduction issue across orthotropic bodies. The equation is shown as

$$\left(a_1 \frac{\partial^2}{\partial x^2} + a_2 \frac{\partial^2}{\partial y^2} \right) T(x, y) = 0 \quad (3.13)$$

$$\text{where } a_1 = \frac{\alpha t_1 a_{22} (a_{11} - a_{12})}{b}, a_2 = \frac{\alpha t_2 a_{11} (a_{22} - a_{12})}{b}, b = a_{11} a_{22} - a_{12}^2 \quad (3.14)$$

Thermal expansion along axis αt_1 and αt_2 are shown.

The coordinate transformation is provided as a simple way to convert (3.13) into a Laplace equation.

$$(\sqrt{a_1})x = \alpha, (\sqrt{a_2})y = \beta \quad (3.15)$$

$$\text{Therefore } \frac{\partial T}{\partial x} = \sqrt{a_1} \frac{\partial T}{\partial \alpha}, \quad \frac{\partial^2 T}{\partial x^2} = a_1 \frac{\partial^2 T}{\partial \alpha^2} \text{ etc.} \quad (3.16)$$

As a result, we get at

$$\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} \right) T(\alpha, \beta) = 0 \quad (3.17)$$

Applying the isotropic body approach yields the solution to the previously described Laplace equation.

3.4 ELASTICITY PROBLEM

The crack-opening solution for isotropic and orthotropic multiply-connected bodies with infinite x and y dimensions and an infinite-strip structure that is finite-width and infinite-length-wise.

Elasticity solutions must be in harmony with the equation of equilibrium.

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} = 0, \quad \frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} = 0 \quad (3.18)$$

Given that the incompatibility formula is and the stress factors are σ_x , $\sigma_{x,y}$, and σ_y .

$$\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2 e_{xx}}{\partial x^2} + \frac{\partial^2 e_{yy}}{\partial y^2} \quad (3.19)$$

e_{xy} , e_y , and e_x represent the components of strain.

In addition to stress-strain relationships,

$$\left. \begin{aligned} e_{xx} &= E^{-1} \{ \sigma_{xx} - \eta e_{yy} \} \\ e_{yy} &= E^{-1} \{ \sigma_{yy} - \eta e_{xx} \} \\ e_{xy} &= \mu^{-1} \sigma_{xy} \end{aligned} \right\} \quad (3.20)$$

Where E, η, μ are the modulus of rigidity of the medium, the Poisson ratio, and the Young's modulus. The equations are for plane-strain conditions of infinite two dimensional body. The solution is obtained through Airy's stress function method (Kushwaha P.S., 1975).

$$\sigma_{xx} = \frac{\partial^2 \phi(x, y)}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi(x, y)}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 \phi(x, y)}{\partial x^2} \quad (3.21)$$

It follows that (3.18) is fulfilled. We get by replacing these in equations (3.19) and (3.20).

$$\left(\nabla^2\right)^2 \phi(x, y) = 0 \quad (3.22)$$

And

$$u_x(x, y) = \frac{2}{\pi} \left(\frac{1+\eta}{E} \right) \int_0^\infty \frac{\sin \xi x}{\xi} \left[(1-\eta) \phi_{,yy} + \eta \xi^2 \phi \right] d\xi \quad (3.23)$$

$$u_y(x, y) = \frac{2}{\pi} \left(\frac{1+\eta}{E} \right) \int_0^\infty \frac{\cos \xi x}{\xi^2} \left[(1-\eta) \phi_{,yyy} + (\eta-2) \xi^2 \phi_{,y} \right] d\xi \quad (3.24)$$

Another way to get the answer to (3.22) is by using the potential function approach. However, the problem is being solved directly as a PDE of the fourth order.

$$\phi(x, y) = \int_0^\infty (A + yB) e^{-\xi y} \cos(\xi x) d\xi \quad (3.25)$$

Equations (3.25), (3.21), (3.23) - (3.26) allow us to easily find the components of stress and displacement. While trying to find a solution for four Griffith fractures that were opened by symmetrical pressures at the crack faces, An issue with mixed-border values and boundary conditions arises when we do this.

$$u_y(x, 0) = 0, \quad 0 \leq |x| \leq b, c \leq |x| \leq d, e \leq |x| < \infty \quad (3.26)$$

$$\sigma_{yy}(x, 0) = \begin{cases} P(x); b < |x| < c \\ Q(x); d < |x| < e \end{cases} \quad (3.27)$$

Equations (3.25) through (3.26) are subjected to geometric and elastic symmetry, which leads to the subsequent quintuple integral equation.

$$\int_0^\infty \phi(\xi) \cos(\xi x) d\xi = 0; 0 \leq x \leq b, c \leq x \leq d, e \leq x < \infty \quad (3.28)$$

$$\int_0^\infty \xi \phi(\xi) \cos(\xi x) d\xi = \begin{cases} P(x); b < x < c \\ Q(x); d < x < e \end{cases} \quad (3.29)$$

A comparable integral equation (IE) to (3.24) - (3.25) can be found above. Thus, the answer is obtainable in that way.

3.5 INFINITE TWO DIMENSSIONAL ORTHOTROPIC BODY

Equilibrium and compatibility relations are (3.18) - (3.19) for an orthotropic body, and the stress-strain relation is

$$\begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \quad (3.30)$$

Using the approach proposed by Kushwaha (1975), we get a constant-coefficient fourth-order equation that may be written as

$$\frac{\partial^4 \phi}{\partial x^4} + 2B_1 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + B_2 \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (3.31)$$

$$\text{with } 2B_1 = \frac{(2a_{12} + a_{66})}{a_{11}}, B_2 = \frac{a_{22}}{a_{11}} \quad (3.32)$$

The answers expressed as components of displacement

$$u_x(x, y) = \frac{2}{\pi} \int_0^\infty \xi^{-1} \sin(\xi x) \left[a_{11} \phi_{,yy} - \xi^2 a_{12} \phi \right] d\xi \quad (3.33)$$

$$u_y(x, y) = \frac{2}{\pi} \int_0^\infty \xi^{-2} \cos(\xi x) \left[a_{11} \phi_{,yyy} - \xi^2 (a_{12} + a_{66}) \phi_{,y} \right] d\xi \quad (3.34)$$

$$\text{With } (r_1 - r_2) \phi(\xi, y) = \{ (r_1 - r_2) A(\xi) - B(\xi) \} e^{-r_1 \xi y} + B(\xi) e^{-r_2 \xi y} \quad (3.35)$$

This is when the variables A and B are two variables that may be defined at will, and the roots of

$$r^4 - 2B_1r^2 + B_2 = 0 \quad (3.36)$$

Upon fulfillment of the mixed-boundary criteria outlined in equations (3.26) through (3.27), the quintuple integral equation (IE) is obtained as

$$\int_0^\infty \phi(\xi) \cos(\xi x) d\xi = 0; \quad 0 \leq x \leq b, c \leq x \leq d, e \leq x < \infty \quad (3.37)$$

$$\int_0^\infty \xi \phi(\xi) \cos(\xi x) d\xi = r_1 \begin{cases} P(x); b < x < c \\ Q(x); d < x < e \end{cases} \quad (3.38)$$

The root of the equation (3.36) is represented by r_1 . We get an integral equation (IE) that is comparable, but with a difference that is a constant multiple of r_1 . The fracture form and normal stress component may be readily determined by knowing the answer of (3.37) - (3.38) $\sigma_{yy}(x, 0)$ for $0 \leq |x| \leq b, c \leq |x| \leq d, e \leq |x| < \infty$ as done in the following chapters.

3.6 ISOTROPIC STRIP

The infinite-length, 2-a-width strip is under consideration. There are four Griffith cracks, arranged symmetrically around the y-axis and running down the breadth of the object. We think of the fissures expanding throughout the fissure faces due to uniform and consistent stresses.

At edges

$$u_x(\pm a, y) = 0; \quad 0 \leq |y| < \infty \quad (3.39)$$

$$\sigma_{xy}(\pm a, y) = 0; 0 \leq |y| < \infty \quad (3.40)$$

$$\sigma_{xy}(x, 0) = 0; 0 \leq |x| \leq a \quad (3.41)$$

with varying border value requirements being

$$u_y(x, 0) = 0; 0 \leq |x| \leq b, c \leq |x| \leq d, e \leq |x| \leq a \quad (3.42)$$

$$\sigma_{yy}(x, 0) = P(x); b < |x| \leq c, d < |x| < e \quad (3.43)$$

Assumedly, the solution to the mixed-boundary issue stated above is

$$u_x(x, y) = \frac{2}{a} \frac{(1 + \eta)}{E} \sum_{n=0}^{\infty} \frac{\sin(\beta_n x)}{\beta_n} \left[(1 - \eta) \phi_{,yy} + \eta \xi^2 \phi \right] d\xi \quad (3.44)$$

$$u_y(x, y) = \frac{2}{a} \left[\phi(0, 0) + \frac{(1 \mp \eta)}{E} \sum_{n=0}^{\infty} \frac{\cos(\beta_n x)}{\beta_n^2} \{ (1 - \eta) \phi_{,yyy} + (\eta - 2) \xi^2 \phi_{,y} \} d\xi \right] \quad (3.45)$$

$$\text{where } \beta_n = \frac{n\pi}{a} = nq.$$

After evaluating u_y from (3.44), (3.45) and (3.20) and satisfying (3.42) (3.43), while (3.39) -(3.41) are satisfied identically. We get quintuple series equation as

$$\frac{1}{2} \phi(0) + \sum_{n=1}^{\infty} \phi(n) \cos(\beta_n x) = 0; 0 \leq x \leq b, c \leq x \leq d, e \leq x \leq a \quad (3.46)$$

$$\sum_{n=1}^{\infty} \beta_n \phi(n) \cos(\beta_n x) = P(x); b < x < c, d < x < e < a \quad (3.47)$$

Equation (3.46) - equation (3.47) may be solved using the technique described by Kushwaha (1979). Our team consists of

$$\phi(n) = \left\langle \int_b^c g(t) + \int_d^e h(t) \right\rangle \frac{\sin(\beta_n t)}{\beta_n} dt \quad (3.48)$$

$$\text{and } \phi(0) = q \left\langle \int_b^c t g(t) dt + \int_d^e t h(t) dt \right\rangle \quad (3.49)$$

combined with series correlation

$$\frac{t}{2} + \sum_{n=1}^{\infty} \frac{\sin(nx) \cos xt}{x} = \begin{cases} \pi/2; x > t \\ \pi/4; x = t \\ 0; x < t \end{cases}$$

satisfy the requirements of (3.46) if

$$\int_b^c g(t) dt = 0, \quad \int_d^e h(t) dt = 0 \quad (3.50)$$

Then substituting (3.48) into (3.49) and using

$$\sum \frac{\sin(\beta_n x) \sin(\beta_n t)}{\beta_n} = \frac{1}{2q} \log \left| \frac{\sin q \left(\frac{x-t}{2} \right)}{\sin q \left(\frac{x+t}{2} \right)} \right|$$

Then using the method of (Kushwaha 1982), we get

$$g(t) = \frac{2}{a^2} \frac{\Delta_1(t)}{\delta(t)}; \quad b < t < c \quad (3.51)$$

$$h(t) = -\frac{2}{a^2} \frac{\Delta_1(t)}{\delta(t)}; \quad d < t < e \quad (3.52)$$

$$\Delta_1(t) = \left[\left\langle \int_b^c - \int_d^e \right\rangle \frac{\sin(qx)P(x)dx}{G(x,t)} + M_1 \cos(qt) + M_2 \right] \quad (3.53)$$

$$\text{with } G(x,t) = \cos(qx) - \cos(qt)$$

$$f(x) = |G(b,x)| |G(x,c)| |G(x,d)| |G(e,x)|$$

By consulting equations (3.50) through (3.52), one may determine the two arbitrary constants M1 and M2.

This paves the way for the determination of other critical physical variables, such the displacement and stress components.

3.7 ISOTROPIC RECTANGLE

An x-axis that contains 2a and a y-axis that follows 2δ are considered in a rectangle with dimensions 2a × 2δ. A total of four Griffith-cracks cover the region.

$y=0, b < |x| < c, d < |x| < e < a$. The boundary conditions are assumed as

$$\sigma_{xy}(\pm a, y) = 0; \quad 0 \leq |y| \leq \delta \quad (3.54)$$

$$u_x(\pm a, y) = 0; \quad 0 \leq |y| \leq \delta \quad (3.55)$$

$$\sigma_{xy}(x, \pm \delta) = 0; \quad 0 \leq |x| \leq a \quad (3.56)$$

$$u_y(x, \pm \delta) = 0; \quad 0 \leq |x| \leq a \quad (3.57)$$

$$\sigma_{xy}(x, 0) = 0; \quad 0 \leq |x| \leq a \quad (3.58)$$

Consistent and consistent force applied to the fracture faces has the potential to open the cracks. Thus

$$u_y(x, 0) = 0; \quad 0 \leq |x| \leq b, c \leq |x| \leq d, e \leq |x| \leq a \quad (3.59)$$

$$\sigma_{yy}(x, 0) = P(x); \quad b < |x| < c, \quad d < |x| < e \quad (3.60)$$

So, the stress components need to be in harmony with the equation of equilibrium (3.18), the compatibility relation (3.19), and (3.20), which all come down to (3.22). It is presumed that the solution across the rectangular domain is

$$u_x(x, y) = \frac{2(1+\eta)}{E} \sum_{n=1}^{\infty} \frac{\sin(\beta_n x)}{\beta_n} \left[(1-\eta)\phi_{,yy} + \eta\beta_n^2 \phi(x, y) \right] d\xi \quad (3.61)$$

$$u_y(x, y) = \frac{2(1+\eta)}{E} \left[\frac{1}{2} u_{yc}(0, y) + \sum_{n=1}^{\infty} \frac{\cos(\beta_n x)}{\beta_n} u_{yc}(\beta_n, y) \right] \quad (3.62)$$

$$u_{yc}(\beta_n, y) = \left[\beta_n \{ (A_n + yB_n) \sinh(\beta_n y) + (C_n + yD_n) \cosh(\beta_n y) \} + (3-2\eta) \{ B_n \cosh(\beta_n y) + D_n \sinh(\beta_n y) \} \right] \quad (3.63)$$

$$\text{and } \phi(x, y) = (A_n + yB_n) \cosh(\beta_n y) + (C_n + yD_n) \sinh(\beta_n y)$$

In this problem, four groups of arbitrary constants need to be determined: A_n , B_n , C_n , and D_n . An analysis of the stress components coupled with a check of the verification of the boundary conditions (3.54) - (3.58) gives us a quintuple-series equation.

$$\frac{D_0}{2} + 2(1-\eta) \sum_{n=1}^{\infty} D_n \tanh(\beta_n \delta) \cos(\beta_n x) = 0 \quad (3.64)$$

$$\text{where } 0 \leq x \leq b, c \leq x \leq d, e \leq x \leq a$$

$$\text{and } \sum_{n=1}^{\infty} \beta_n D_n M(\beta_n \delta) \cos(\beta_n x) = P(x) \quad (3.65)$$

$$\text{where } b < x < c, \quad d < x < e$$

$$\text{with } M(\beta_n \delta) = \frac{\sinh(2\beta_n \delta) + 2\beta_n \delta}{\sinh(2\beta_n \delta)} \quad (3.66)$$

Following that, a basic multiplication

$$\left. \begin{aligned} D_n \tanh(\beta_n \delta) &= \phi_n \\ D_0 &= \phi_0 \end{aligned} \right\} \quad (3.67)$$

Using (3.67) in (3.64) – (3.65), we get

$$\frac{\phi_0}{2} + \sum_{n=1}^{\infty} \phi(n) \cos \beta_n x = 0, 0 \leq x \leq b, c \leq x \leq d, e \leq x \leq a \quad (3.68)$$

$$\sum_{n=1}^{\infty} \beta_n \phi_n \cos(\beta_n x) = N(x), b < x < c, d < x < e \quad (3.69)$$

$$\text{with } N(x) = P(x) - \sum_{n=1}^{\infty} \beta_n \phi(n) M(\beta_n \delta) \cos(\beta_n x)$$

So, using the same approach as in relation (3.5), we can easily derive the solutions of series relations (3.64) and (3.65).

So, it is clear that determining the potential function over Multi-connected isotropic or orthotropic infinite bodies can be solved, in the case of four Griffith cracks, to the quintuple-integration solution, and this can be done by the method of (Kushwaha. 1982).

Regarding the elasticity issue, this boils down to the same quintuple integral equation (IE) as for the potential function: how to obtain the components of stress and displacement across a multiply-connected (with four Griffith cracks present) isotropic or orthotropic material. In two different configurations, a finite rectangle with dimensions $2a$ by 2δ and an infinite strip with width $2a$ by 2δ , the process of determining the stress and displacement components over a multi-connected body with four Griffith cracks as shown in Kushwaha (1982) reduces to a quintuple series equation. Closed form representations of stress and displacements will be explained and reported in problems 1, 2, 3, and 4. Also, problems 5 and 6 demonstrate the real determination of potential function across isotropic and orthotropic media.

RESULT AND DISCUSSION

Many intelligent gadgets are available nowadays. The smart gadgets' efficacy is drastically reduced when they develop cracks. The possibility that fluctuations in temperature may impact the overall functioning of smart devices was raised by someone. Among other temperature-dependent material characteristics, this article discusses the impact of electric current loading on stress, temperature, and a broken metal plate. We are also talking about the potential for non-stationary electrical currents to break plates. We used a single Cauchy kernel to solve the issue after simplifying it to one system of integral equations.

By using the Fourier transform, the problem with the dual integral equation (IE) was resolved (Sneddon, 1946). An equation system including three integrals was resolved by Lowengrub and Srivastava. The quintuple integral problem was also solved by (Kushwaha, 1975).

Each issue in the chapter will begin with its source and the method for fixing it, according to the chapter's design. The problems are formulated after the introduction. Then you'll be able to solve any integral or series problem. As a result of solving the quintuple integral equation, the physical quantities relevant to fracture mechanics may be seen. Each problem ends with a discussion of any special instances that may arise.

There are many different types of fractures, and each one has its own specific geometry. Asymmetry in the forces at work resulted in the crack opening being sold. The extension method was used to extend 2 Griffith-cracks. Although the method developed by Kushwaha and Awasthi (1975) was successful in resolving these challenges, the method should be used here.

4.1 FRACTURE BEHAVIOR OF FOUR GRIFFITH CRACKS UNDER SYMMETRICAL CRACK-FACE FORCES IN AN INFINITE ELASTIC BODY

4.1.1 Introduction

Problems with isotropic multiply-connected infinite media and planar strain were considered in the present investigation. The problem evolved into a mixed boundary value problem mathematically when the four Griffith fractures were made accessible, and the medium multiplied. This happened due to the fact that crack face pressures can be seen in the rest of the part of boundary dispositions.

These fissures may take up a lot of space in the infinite isotropic media, $y = 0, b < |x| < c, d < |x| < e < \infty$. So the inner pair is equal in length, and spaced at equal distances in respect to the y-axis along the x-axis, $y = 0, d < |x| < e < \infty$. The external pair are of the same length, and are at different positions. Asymmetrical forces at crack faces affect four Griffith-cracks. Given the mixed boundary value problem:

$$\sigma_{xy}(x, 0^\pm) = \begin{cases} q_1^\pm(x), & x \in I_1^\pm \\ q_2^\pm(x), & x \in I_2^\pm \\ q_3^\pm(x), & x \in I_3^\pm \\ q_4^\pm(x), & x \in I_4^\pm \end{cases} \quad (4.1)$$

$$\sigma_{yy}(x, 0^\pm) = \begin{cases} p_1^\pm(x), & x \in I_1^\pm \\ p_2^\pm(x), & x \in I_2^\pm \\ p_3^\pm(x), & x \in I_3^\pm \\ p_4^\pm(x), & x \in I_4^\pm \end{cases} \quad (4.2)$$

$$\text{where, } I_1^\pm = (b, c), I_2^\pm = (-c, -b), I_3^\pm = (d, e), I_4^\pm = (-e, -d) \quad (4.3)$$

Since $(\pm)$ over quantities are dependent on $y = 0^+$ and $y = 0^-$, continuity conditions are met at $y = 0$ at all boundaries except those of crack faces.

$$u_x(x, 0^+) = u_x(x, 0^-), \quad u_y(x, 0^+) = u_y(x, 0^-) \quad (4.4)$$

$$\sigma_{xy}(x, 0^+) = \sigma_{xy}(x, 0^-), \quad \sigma_{yy}(x, 0^+) = \sigma_{yy}(x, 0^-) \quad (4.5)$$

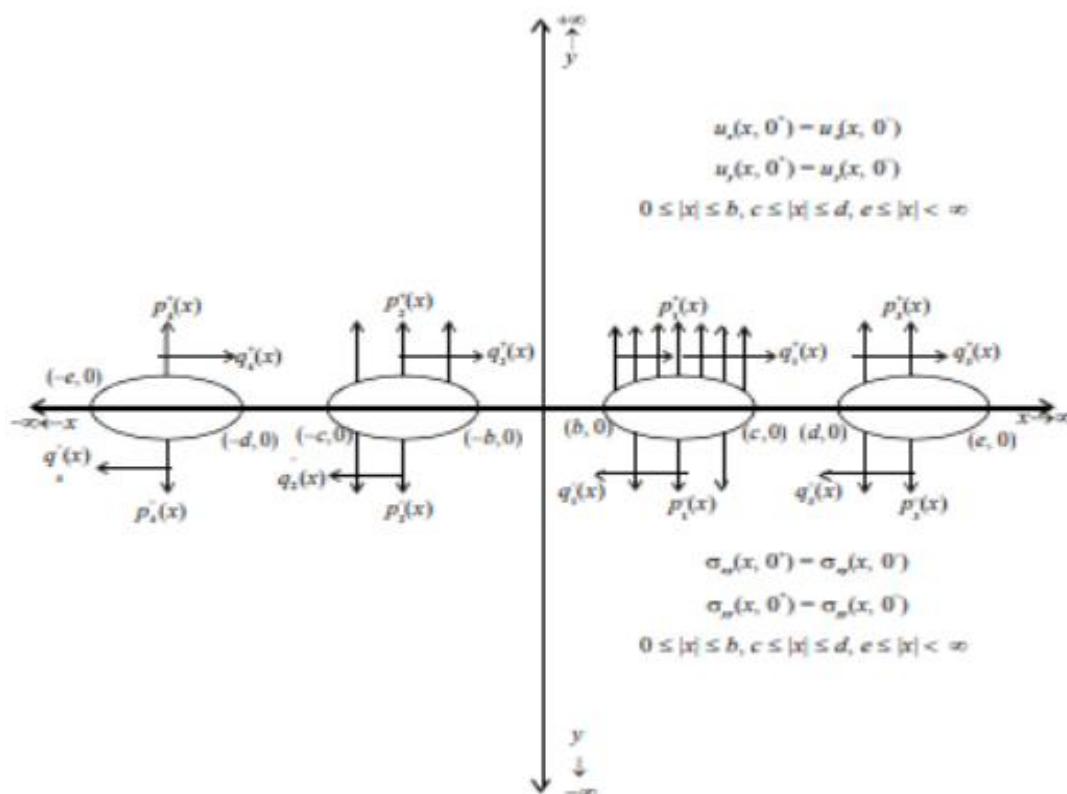


Figure 4.1: Asymmetrical forces at fracture faces in isotropic infinite media cause crack opening geometry.

For $0 \leq |x| \leq b$, $c \leq |x| \leq d$, $e \leq |x| < \infty$, (u_x, u_y) and $(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$ are

Stress components and displacement at the medium's general point (x, y) . Furthermore,

it is presumed that all material amounts disappear as $\sqrt{x^2 + y^2} \rightarrow \infty$. We checked throughout that, see (Burniston, 1969).

We checked throughout that, see (Burniston, 1969).

Cracks are opening. The Fourier transform is,

$$f_{c,s}(\xi) = \int_0^\infty f(x) \{\cos \xi x, \sin \xi x\} dx \quad (4.6)$$

With usual inversion formula.

4.1.2 Formulation

The method proposed by Sneddon in 1946 provides a solution to the present issue with linear elasticity. We classify the physical quantities at any given place in the medium as either symmetric or anti-symmetric, depending on whether they are

$$\sigma_{ij}(x, y) = \sigma_{ij}^{s_k}(x, y) + \sigma_{ij}^{a_k}(x, y), \quad i, j = x, y \quad (4.7)$$

$$u_i(x, y) = u_i^{s_k}(x, y) + u_i^{a_k}(x, y), \quad i = x, y \quad (4.8)$$

And $k = 3, 4$ indicate $y > 0$ and $y < 0$.

s_k and a_k are the symmetric and anti-symmetric sides of the problem, respectively. The list of potential solutions. $(-\infty, \infty) \cup (-\infty, \infty)$, i.e., the whole two-dimensional plane, is transformed into $[0, \infty) \cup [0, \infty)$ i.e. $x \geq 0, y = 0$. Now, we assume the displacement components as,

$$u_x^{s_k}(x, y) = \alpha_0 \int_0^\infty \frac{\sin(\xi x)}{\xi} \left[(1 - \eta) \frac{\partial^2 G^{s_k}}{\partial y^2} + \eta \xi^2 G^{s_k} \right] d\xi \quad (4.9)$$

$$u_x^{a_k}(x, y) = \alpha_0 \int_0^\infty \frac{\cos(\xi x)}{\xi} \left[(1 - \eta) \frac{\partial^2 G^{a_k}}{\partial y^2} + \eta \xi^2 G^{a_k} \right] d\xi \quad (4.10)$$

$$u_y^{s_k}(x, y) = \alpha_0 \int_0^\infty \frac{\cos(\xi x)}{\xi^2} \left[(1 - \eta) \frac{\partial^3 G^{s_k}}{\partial y^3} + (\eta + 2) \xi^2 \frac{\partial G^{s_k}}{\partial y} \right] d\xi \quad (4.11)$$

$$\left. \begin{aligned} u_y^{a_k}(x, y) &= \alpha_0 \int_0^\infty \frac{\sin(\xi x)}{\xi^2} \left[(1 - \eta) \frac{\partial^3 G^{a_k}}{\partial y^3} + (\eta + 2) \xi^2 \frac{\partial G^{a_k}}{\partial y} \right] d\xi, \\ \alpha_0 &= \frac{2(1 + \eta)}{\pi E} \end{aligned} \right\} \quad (4.12)$$

$$G^{S_k}(\xi, y) = (A_{k-2} + yB_{k-2})e^{\mp \xi y} \quad (4.13)$$

$$G^{a_k}(\xi, y) = (C_{k-2} + yD_{k-2})e^{\mp \xi y} \quad (4.14)$$

“For $k = 3, 4$, which means $y > 0$ and $y < 0$, use the operators $(-)$ and $(+)$. If this displacement factor is assumed, then equations requiring compatibility, equilibrium, continuity, and the like may be solved. This is Airy's stress function, and its sign is G . Let η be the Poisson ratio of the medium and E be its Young's modulus. Finding 8 random constants in (4.13)–(4.14) fixes it. Applying (4.9)–(4.14) and the stress-strain relation yields stress components.”

$$\sigma_{xy}^{S_k}(x, y) = \mp \int_0^\infty \xi \sin(\xi x) [\xi A_{k-2} + B_{k-2}(\xi y - 1)] e^{\mp \xi y} d\xi \quad (4.15)$$

$$\sigma_{xy}^{a_k}(x, y) = \pm \int_0^\infty \xi \cos(\xi x) [\xi C_{k-2} + D_{k-2}(\xi y - 1)] e^{\mp \xi y} d\xi \quad (4.16)$$

$$\sigma_{yy}^{S_k}(x, y) = - \int_0^\infty \xi^2 \cos(\xi x) [A_{k-2} + yB_{k-2}] e^{\mp \xi y} d\xi \quad (4.17)$$

$$\sigma_{yy}^{a_k}(x, y) = - \int_0^\infty \xi^2 \sin \xi x [C_{k-2} + yD_{k-2}] e^{\mp \xi y} d\xi \quad (4.18)$$

$$\sigma_{xx}^{S_k}(x, y) = \int_0^\infty \xi \cos(\xi x) [\xi (A_{k-2} + yB_{k-2}) - 2B_{k-2}] e^{\mp \xi y} d\xi \quad (4.19)$$

$$\sigma_{xx}^{a_k}(x, y) = \int_0^\infty \xi \cos(\xi x) [\xi (C_{k-2} + yD_{k-2}) - 2D_{k-2}] e^{\mp \xi y} d\xi \quad (4.20)$$

4.1.3 Reduction to Quintuple Integral Equations

With the separations of continuity conditions as symm and anti-symm, continuity of physical quantities is satisfied too. The continuity conditions (4.20) and using boundary conditions (4.1) – (4.2). The result is obtained by inverting the Fourier integral transform.

Following this, we get by inverting the Fourier integral transform:

$$A_1 - A_2 = \xi^{-2} \left[\left\langle \int_b^c p_1^{e_1}(x) + \int_d^e p_1^{e_2}(x) \right\rangle \cos(\xi x) dx \right] \quad (4.21)$$

$$C_1 - C_2 = \xi^{-2} \left[\left\langle \int_b^c p_1^{0_1}(x) + \int_d^e p_1^{0_2}(x) \right\rangle \sin(\xi x) dx \right] \quad (4.22)$$

$$\xi(A_1 + A_2) - (B_1 - B_2) = \xi^{-1} \left[\left\langle \int_b^c Q_1^{e_1}(x) + \int_d^e Q_1^{e_2}(x) \right\rangle \sin(\xi x) dx \right] \quad (4.23)$$

$$\xi(C_1 + C_2) - (D_1 - D_2) = \xi^{-1} \left[\left\langle \int_b^c Q_1^{0_1}(x) + \int_d^e Q_1^{0_2}(x) \right\rangle \cos(\xi x) dx \right] \quad (4.24)$$

Where

$$\left. \begin{aligned} p_1^{e_1}(x) &= p_1^+ + p_2^+ - (p_1^- + p_2^-) \\ p_1^{e_2}(x) &= p_3^+ + p_4^+ - (p_3^- + p_4^-) \end{aligned} \right\} \quad (4.25)$$

$$\left. \begin{aligned} p_1^{0_1}(x) &= p_1^+ - p_2^+ - (p_1^- - p_2^-) \\ p_1^{0_2}(x) &= p_3^+ - p_4^+ - (p_3^- - p_4^-) \end{aligned} \right\} \quad (4.26)$$

$$\left. \begin{aligned} Q_1^{e_1}(x) &= q_1^- - q_2^- - (q_1^+ - q_2^+) \\ Q_1^{e_2}(x) &= q_3^- - q_4^- - (q_3^+ - q_4^+) \end{aligned} \right\} \quad (4.27)$$

$$\left. \begin{aligned} Q_1^{0_1}(x) &= q_1^- + q_2^- - (q_1^+ + q_2^+) \\ Q_1^{0_2}(x) &= q_3^- + q_4^- - (q_3^+ + q_4^+) \end{aligned} \right\} \quad (4.28)$$

Thus, four equations (4.21–4.24) assist determine four constants. Rests are dictated by mixed-boundary circumstances. Now we employ continuity criteria of displacement components (4.4)– (4.5) and partition them into symmetric and anti-symmetric parts. So using (4.4) – (4.5) and (4.9) – (4.12) and (4.21) – (4.26), we obtain

$$\int_0^\infty (B_1 - B_2) \cos(\xi x) d\xi = t_1 \begin{cases} F_{11}(x), & x \in I_1 \\ F_{12}(x), & x \in I_3 \\ F_{13}(x), & x \in I_5 \end{cases} \quad (4.29)$$

$$\int_0^\infty (D_1 - D_2) \sin(\xi x) d\xi = t_1 \begin{cases} F_{21}(x), & x \in I_1 \\ F_{22}(x), & x \in I_3 \\ F_{23}(x), & x \in I_5 \end{cases} \quad (4.30)$$

$$\int_0^\infty (B_1 + B_2) \sin(\xi x) d\xi = t_1 \begin{cases} F_{31}(x), & x \in I_1 \\ F_{32}(x), & x \in I_3 \\ F_{33}(x), & x \in I_5 \end{cases} \quad (4.31)$$

$$\int_0^\infty (D_1 + D_2) \cos(\xi x) d\xi = t_1 \begin{cases} F_{41}(x), & x \in I_1 \\ F_{42}(x), & x \in I_3 \\ F_{43}(x), & x \in I_5 \end{cases} \quad (4.32)$$

With

$$\left. \begin{aligned} t_1 &= [2(1 + \eta) / E]^{-1}, F_{13} = F_{21} = F_{31} = F_{43} = 0, \\ I_1 &= [0, b], I_3 = [c, d], I_5 = [e, \infty) \end{aligned} \right\} \quad (4.33)$$

$$\left. \begin{aligned} F_{11} &= \frac{\pi}{2} \left\langle \int_b^c Q_1^{e_1}(x) dx + \int_d^e Q_1^{e_2}(x) dx \right\rangle, F_{12} = \frac{\pi}{2} \int_d^e Q_1^{e_2}(x) dx, \\ F_{22} &= \frac{\pi}{2} \int_b^c Q_1^{0_1}(x) dx, F_{23} = \frac{\pi}{2} \int_d^e Q_1^{0_2}(x) dx, \\ F_{32} &= \frac{\pi}{2} \int_b^c p_1^{e_1}(x) dx, F_{33} = \frac{\pi}{2} \int_d^e p_1^{e_2}(x) dx + F_{32} \\ F_{41} &= \frac{\pi}{2} \int_b^c p_1^{0_1}(x) dx + F_{42}, F_{42} = \frac{\pi}{2} \int_d^e p_1^{0_2}(x) dx \end{aligned} \right\} \quad (4.34)$$

Now we evaluate $(\sigma_{yy}^{(3)} + \sigma_{yy}^{(4)})$ and $(\sigma_{xy}^{(3)} + \sigma_{xy}^{(4)})$ Applying the formulas (4.15)–(4.18) to (4.21)–(4.24) for both symmetric and antisymmetric cases yields.

$$\int_0^\infty \xi(B_1 - B_2) \cos(\xi x) d\xi = \begin{cases} P_{11} + \int_b^c \frac{y Q_1^{e_1}(y) dy}{y^2 - x^2}, x \in I_2 \\ P_{12} + \int_b^e \frac{y Q_1^{e_2}(y) dy}{y^2 - x^2}, x \in I_4 \end{cases} \quad (4.35)$$

$$\int_0^\infty \xi(D_1 - D_2) \sin(\xi x) d\xi = \begin{cases} P_{21} + x \int_b^c \frac{Q_1^{0_1}(y) dy}{y^2 - x^2}, x \in I_2 \\ P_{22} + x \int_b^e \frac{Q_1^{0_2}(y) dy}{y^2 - x^2}, x \in I_4 \end{cases} \quad (4.36)$$

$$\int_0^\infty \xi(B_1 + B_2) \sin(\xi x) d\xi = \begin{cases} Q_{11} + x \int_b^c \frac{P_1^{e_1}(y) dy}{y^2 - x^2}, x \in I_2 \\ Q_{12} + x \int_d^e \frac{P_1^{e_2}(y) dy}{y^2 - x^2}, x \in I_4 \end{cases} \quad (4.37)$$

$$\int_0^\infty \xi(D_1 + D_2) \cos(\xi x) d\xi = \begin{cases} Q_{21} + \int_b^c \frac{P_1^{0_1}(y) dy}{y^2 - x^2}, x \in I_2 \\ Q_{22} + \int_d^e \frac{P_1^{0_2}(y) dy}{y^2 - x^2}, x \in I_4 \end{cases} \quad (4.38)$$

Where

$$\left. \begin{aligned} I_2 &= (b, c), I_4 = (d, e) \quad \text{and} \\ P_{11} &= p_1^+ + p_1^- + p_2^+ + p_2^- \\ P_{12} &= p_3^+ + p_3^- + p_4^+ + p_4^- \\ P_{21} &= p_1^+ - p_1^- + (p_2^+ - p_2^-) \\ P_{22} &= p_3^+ - p_3^- + (p_4^+ - p_4^-) \\ Q_{11} &= q_1^+ + q_1^- + q_2^+ + q_2^- \\ Q_{12} &= q_3^+ + q_3^- + q_4^+ + q_4^- \\ Q_{21} &= q_1^+ - q_1^- + q_2^+ - q_2^- \\ Q_{22} &= q_3^+ - q_3^- + q_4^+ - q_4^- \end{aligned} \right\} \quad (4.39)$$

Thus rest four constants $B_1, B_2; D_1, D_2$, may be ascertained by solving the quintuple integral equations (IE) provided by (4.30) - (4.33) and (4.36) - (4.39).

4.1.4 The Quintuple Integral equations (IE) and Their Solution

See (Kushwaha P.S., 1975) for the assumed trial solution of the quintuple integral equations.

$$B_1 - B_2 = \frac{2}{\pi \xi} \left[\left\langle \int_b^c g_{11}(t) + \int_d^e g_{12}(t) \right\rangle \sin(\xi t) dt - t_1 \left\langle \int_0^b F'_{11}(t) + \int_c^d F'_{12}(t) \right\rangle \sin(\xi t) dt \right] \quad (4.40)$$

$$D_1 - D_2 = \frac{2}{\pi \xi} \left[\left\{ \left\langle \int_b^c g_{21}(t) + \int_d^e g_{22}(t) \right\rangle - t_1 \left\langle \int_c^d F'_{22}(t) + \int_e^\infty F'_{23}(t) \right\rangle \right\} (1 - \cos \xi t) dt \right] \quad (4.41)$$

$$B_1 + B_2 = \frac{2}{\pi\xi} \left[\left\langle \int_b^c g_{31}(t) + \int_d^e g_{32}(t) \right\rangle - t_1 \left\langle \int_c^d F'_{32}(t) + \int_e^\infty F'_{33}(t) \right\rangle \right] (1 - \cos \xi t) dt \quad (4.42)$$

$$D_1 + D_2 = \frac{2}{\pi\xi} \left[\left\langle \int_b^c g_{41}(t) + \int_d^e g_{42}(t) \right\rangle - t_1 \left\langle \int_c^d F'_{41}(t) + \int_e^\infty F'_{42}(t) \right\rangle \right] \sin(\xi t) dt \quad (4.43)$$

The above assumption (4.40) – (4.43) satisfies the equations (4.29) – (4.32) identically if

$$\int_b^c g_{11}(t) dt = t_1 \langle F_{12}(d) - F_{11}(b) \rangle \quad (4.44)$$

$$\int_d^e g_{12}(t) dt = t_1 F_{12}(d)$$

$$\int_b^c g_{21}(t) dt = -t_1 F_{12}(d) \quad (4.45)$$

$$\int_d^e g_{22}(t) dt = t_1 \langle F_{22}(d) - F_{23}(e) \rangle$$

$$\int_b^c g_{31}(t) dt = -t_1 F_{23}(c) \quad (4.46)$$

$$\int_d^e g_{32}(t) dt = t_1 \langle F_{32}(d) - F_{33}(e) \rangle$$

$$\int_b^c g_{41}(t)dt = t_1 \langle F_{42}(c) - F_{41}(b) \rangle \quad (4.47)$$

$$\int_d^e g_{42}(t)dt = -t_1 F_{42}(d)$$

The values of F_{ij} are provided in section 4.34. We get by inverting the integrals according to the methods of (Kushwaha, 1975) and substituting (4.40) - (4.43) into (4.35) - (4.38).

$$g_{11}(t) = \frac{\Delta_1(t)}{\pi^2 \theta(t)}, \quad t \in I_2 \quad (4.48)$$

$$\text{with } \theta(t) = \left\{ (t^2 - b^2)(c^2 - t^2)(d^2 - t^2)(e^2 - t^2) \right\}^{1/2}$$

$$\begin{aligned} \Delta_1(t) = & \left\langle \int_b^c T_{11} - \int_d^e T_{12} \right\rangle \frac{y \theta(y) dy}{y^2 - t^2} \\ & + t_1 \left\langle \int_c^d F'_{12} + \int_e^\infty F'_{13} \right\rangle \frac{y \theta(y)}{y^2 - t^2} + t^2 L_1 + L_2 \end{aligned} \quad (4.49)$$

$$\begin{aligned} \Delta_2(t) = & \left\langle \int_b^c T_{21} - \int_d^e T_{22} \right\rangle \frac{\theta(y) dy}{y^2 - t^2} \\ & + t_1 \left\langle \int_c^d F'_{22} - \int_e^\infty F'_{23} \right\rangle \frac{\theta(y) dy}{y^2 - t^2} + t^2 M_1 + M_2 \end{aligned} \quad (4.50)$$

$$\begin{aligned} \Delta_3(t) = & \left\langle \int_b^c T_{31} - \int_d^e T_{32} \right\rangle \frac{\theta(y) dy}{y^2 - t^2} \\ & + t_1 \left\langle \int_c^d F'_{32} - \int_e^\infty F'_{33} \right\rangle \frac{\theta(y) dy}{y^2 - t^2} + t^2 R_1 + R_2 \end{aligned} \quad (4.51)$$

$$\begin{aligned} \Delta_4(t) = & \left\langle \int_b^c T_{41} - \int_d^e T_{42} \right\rangle \frac{\theta(y) dy}{y^2 - t^2} \\ & + t_1 \left\langle \int_c^d F'_{42} + \int_e^\infty F'_{43} \right\rangle \frac{\theta(y) dy}{y^2 - t^2} + t^2 N_1 + N_2 \end{aligned} \quad (4.52)$$

where , , , , 1,2 L M N R i = are constants to be determined through (4.48) and (4.44) – (4.47). And,

$$\left. \begin{aligned} T_{11}(x) &= P_{11}(x) - \int_b^c \frac{y Q_1^e(y) dy}{y^2 - x^2} \\ T_{12} &= P_{12}(x) - \int_d^e \frac{y Q_1^{e_2}(y) dy}{y^2 - x^2} \end{aligned} \right\} \quad (4.53)$$

$$\begin{aligned} T_{21}(x) &= P_{21}(x) - \int_b^c \frac{x Q_1^{0_1} dy}{y^2 - x^2} \\ T_{22}(x) &= P_{12}(x) - \int_d^e \frac{x Q_1^{0_2} dy}{y^2 - x^2} \end{aligned} \quad (4.54)$$

$$\left. \begin{aligned} T_{31}(x) &= Q_{11}(x) + x \int_b^c \frac{P_1^{e_1} dy}{y^2 - x^2} \\ T_{32}(x) &= Q_{12}(x) + x \int_d^e \frac{P_1^{e_2} dy}{y^2 - x^2} \end{aligned} \right\} \quad (4.55)$$

$$\left. \begin{aligned} T_{41}(x) &= Q_{21}(x) + \int_b^c \frac{y P_1^{0_1} dy}{y^2 - x^2} \\ T_{42}(x) &= Q_{22}(x) + \int_d^e \frac{y P_1^{0_2} dy}{y^2 - x^2} \end{aligned} \right\} \quad (4.56)$$

What follows is an analysis of the physical variables.

4.1.5 Physical Measurements

The most important two physical characteristics in fracture mechanics include stress intensity at the end of the crack and the crack opening displacement.

Stress-Components

Assessment of stress components is done for $y \geq 0$;

$x \in [0, b] \cup [c, d] \cup [e, \infty)$. The normal component of stress is $\sigma_{yy}(x, 0)$ and shear stress is $\sigma_{xy}(x, 0)$.

Normal Stress

It is possible to find the typical stress for symm and anti-symm components using integrals (4.35) and (4.36), respectively $x \in I_1 \cup I_3 \cup I_5$.

$$\sigma_{yy}^{s_3}(x, 0) = \sigma_{yy}^{s_4}(x, 0), \quad x \in I_1 \cup I_3 \cup I_5$$

$$\sigma_{yy}^{s_3}(x, 0) = \frac{2}{\pi} \left[\left\{ \int_b^c g_{11}(t) + \int_d^e g_{12}(t) - t_1 \left\langle \int_c^d F'_{12}(t) \right\rangle \right\} \frac{t dt}{t^2 - x^2} + \frac{\pi}{2} F_1(x) \right],$$

$x \in I_1 \cup I_3 \cup I_5$ (4.57)

With

$$F_1(x) = \begin{cases} \int_b^c Q_1^{e_1}(x) dx + \int_d^e Q_1^{e_2}(x) dx, & x \in I_1 \\ \int_d^e Q_1^{e_2}(x) dx, & x \in I_3 \\ 0, & x \in I_5 \end{cases}$$

(4.58)

$$\sigma_{yy}^{a_3}(x, 0) = \frac{1}{2\pi} \left[\left[\int_b^c g_{21}(t) + \int_d^e g_{22}(t) - t_1 \left\langle \int_c^d F'_{22}(t) + \int_e^\infty F_{23}(t) \right\rangle \right] \frac{t dt}{t^2 - x^2} + \frac{\pi}{2} F_2(x) \right],$$

$x \in I_1 \cup I_3 \cup I_5$

$$\sigma_{yy}^{a_3}(x, 0) = \sigma_{yy}^{a_4}(x, 0)$$

(4.59)

$$F(x) = \begin{cases} \int_b^c Q_1^{0_1} dt + \int_d^e Q_1^{0_2} dt, & x \in I_1 \\ \int_d^e Q_1^{0_2}(t) dt, & x \in I_3 \\ 0, & x \in I_5 \end{cases} \quad (4.60)$$

Shear Stress

The component of shear $\sigma_{xy}(x, 0^+) = \sigma_{xy}(x, 0^-)$ for $x \in I_1 \cup I_3 \cup I_5$ is evaluated through the value of integrals of left hand side of (4.38) (4.39) which is given as,

$$\sigma_{xy}^{s_3}(x, 0) = \sigma_{xy}^{s_4}(x, 0) = \frac{x}{2\pi} \left[\left\{ \int_b^c g_{31}(t) + \int_d^e g_{32}(t) + t_1 \left\langle \int_c^d F_{32}(t) + \int_c^\infty F'_{33}(t) \right\rangle \right\} \frac{dt}{t^2 - x^2} + \frac{\pi}{2} F_3(x) \right], \quad x \in I_1 \cup I_3 \cup I_5 \quad (4.61)$$

$$F_3(x) = \begin{cases} 0, & x \in I_1 \\ \int_b^c P_1^{e_1}(y) dy, & x \in I_3 \\ \int_b^c P_1^{e_1}(y) dy + \int_d^e P_1^{e_2} dy, & x \in I_5 \end{cases} \quad (4.62)$$

And

$$\sigma_{xy}^{a_3}(x, 0) = -\frac{1}{2\pi} \left[\left\{ \int_b^c g_{41}(t) + \int_d^e g_{42}(t) dt + t_1 \left\langle \int_c^d F'_{42}(t) + \int_e^\infty F'_{43}(t) \right\rangle \right\} \frac{tdt}{t^2 - x^2} \right] + \frac{\pi}{2} F_4(x)$$

$$\sigma_{xy}^{a_3}(x, 0) = \sigma_{xy}^{a_4}(x, 0) \quad (4.63)$$

$$F_4(x) = \begin{cases} 0, & x \in I_1 \\ \int_b^c P_1^{0_1}(y) dy, & x \in I_3 \\ \int_b^c P_1^{0_1}(y) dy + \int_d^e P_1^{0_2}(y) dy, & x \in I_5 \end{cases} \quad (4.64)$$

After obtaining the values of $g_{ij}(t)$ from (4.48) and evaluating the integrals, we have,

$$\sigma_{yy}^{s_3}(x,0) = \begin{cases} \frac{\Delta_1(x)}{2\pi\theta(x)} + F_1(x), x \in I_1 \\ -\frac{\Delta_1(x)}{2\pi\theta(x)} + F_1(x), x \in I_3 \\ \frac{\Delta_1(x)}{2\pi\theta(x)} + F_1(x), x \in I_5 \end{cases} \quad (4.65)$$

Where $\theta(x)$ and $\Delta_1(x)$ are given by (4.60) and

$$\sigma_{yy}^{a_3}(x,0) = \begin{cases} \frac{x}{2\pi} \frac{\Delta_2(x)}{\theta(x)} + \frac{\pi}{4} F_2(x), x \in I_1 \\ -\frac{x}{2\pi} \frac{\Delta_2(x)}{\theta(x)} + \frac{\pi}{4} F_2(x), x \in I_3 \\ \frac{x}{2\pi} \frac{\Delta_2(x)}{\theta(x)} + \frac{\pi}{4} F_2(x), x \in I_5 \end{cases} \quad (4.66)$$

Where $\Delta_2(x)$, $F_2(x)$ are given by (4.54) & (4.64), respectively

$$\sigma_{xy}^{s_3}(x,0) = \begin{cases} \frac{x}{2} \frac{\Delta_3(x)}{\theta(x)} + \frac{\pi}{2} F_3(x), x \in I_1 \\ -\frac{x}{2} \frac{\Delta_3(x)}{\theta(x)} + \frac{\pi}{4} F_3(x), x \in I_3 \\ \frac{x}{2} \frac{\Delta_3(x)}{\theta(x)} + \frac{\pi}{4} F_3(x), x \in I_5 \end{cases} \quad (4.67)$$

Where $\Delta_3(x)$ and $F_3(x)$ are given by (4.55) & (4.63), respectively.

$$\sigma_{xy}^{a_4}(x, 0) = \begin{cases} \frac{\Delta_4(x)}{2\pi \theta(x)} + F_4(x), x \in I_1 \\ -\frac{\Delta_4(x)}{2\pi \theta(x)} + F_4(x), x \in I_3 \\ \frac{\Delta_4(x)}{2\pi \theta(x)} + F_4(x), x \in I_5 \end{cases} \quad (4.68)$$

Where $\Delta_4(x)$ and $F_4(x)$ are given by (4.56) & (4.68) respectively.

Displacement of Crack Opening

A fracture form is seen when the COD is evaluated. COD will be calculated by calculating certain integrals and the connection (4.40) - (4.43), in addition to the fractions on the left side of the equations (4.29) through (4.30).

$$u_y^{s_3}(x, 0) - u_y^{s_4}(x, 0) = (1 - \eta)\alpha_0 \begin{cases} \int_x^c g_{11}(t) + t_1 F_{12}(d) dt - U_1(x), x \in I_2 \\ \int_x^e g_{12}(t) dt - U_1(x), x \in I_4 \end{cases} \quad (4.69)$$

$$U_1(x) = \frac{\pi}{4} \left[\int_b^c p_1^{e_1}(t) + \int_a^e p_1^{e_2}(t) \right] \log |t^2 - x^2| dt \quad (4.70)$$

$$u_y^{s_3}(x, 0) + u_y^{s_4}(x, 0) = \alpha_0 \begin{cases} \int_x^c g_{11}(t) dt - t_1(1 - \eta)g_{31}(x) - t_1(F_{12}(d) - F_{12}(c)), x \in I_2 \\ \int_x^e g_{12}(t) dt - t_1(1 - \eta)g_{32}(x), x \in I_4 \end{cases} \quad (4.71)$$

So, we can easily find using (4.33) - (4.34) $u_y^{s_3}$ and $u_y^{s_4}$.

Factors Influencing Stress Levels

As a result of crack tips, the S.I.F. is

$$\left. \begin{aligned} [K_b, N_b] &= \lim_{x \rightarrow b^-} \sqrt{b-x} [\sigma_{yy}, \sigma_{xy}]; & [K_c, N_c] &= \lim_{x \rightarrow c^+} \sqrt{x-c} [\sigma_{yy}, \sigma_{xy}] \\ [K_d, N_d] &= \lim_{x \rightarrow d^-} \sqrt{d-x} [\sigma_{yy}, \sigma_{xy}]; & [K_e, N_e] &= \lim_{x \rightarrow e^+} \sqrt{x-e} [\sigma_{yy}, \sigma_{xy}] \end{aligned} \right\} \quad (4.72)$$

4.1.6 Special Type of Loading

To optimize the analysis and approach based on the integral equation, we consider special loads over fracture faces.

$$p_1^\pm = p_2^\pm = p_3^\pm = p_4^\pm = p_0 \quad (4.73)$$

$$q_1^\pm = q_2^\pm = q_3^\pm = q_4^\pm = 0 \quad (4.74)$$

The implication is that cracks are expanding due to a continuous and consistent force acting on the fracture faces. In (4.25) (4.28), we use equations (4.73) -(4.74) to get

$$\left. \begin{aligned} p_1^{\varepsilon_1}(x) &= 0, p_1^{\varepsilon_2}(x) = 0, p_1^{0_1}(x) = 0, p_1^{0_2}(x) = 0 \\ Q_1^{\varepsilon_1}(x) &= 0 = Q_1^{\varepsilon_2}(x) = Q_1^{0_1}(x) = Q_1^{0_2}(x) \end{aligned} \right\} \quad (4.75)$$

Now we use (4.75) in (4.34) we get,

$$\left. \begin{aligned} F_{11}(x) &= F_{12}(x) = F_{13}(x) = 0, F_{21}(x) = F_{22}(x) = F_{23}(x) = 0 \\ F_{31}(x) &= F_{32}(x) = F_{33}(x) = 0, F_{41}(x) = F_{42}(x) = F_{43}(x) = 0 \end{aligned} \right\} \quad (4.76)$$

Now using (4.75) in (4.39) we get

$$\left. \begin{aligned} P_{11} &= 4p_0 = P_{12}, P_{21} = P_{22} = 0 \\ Q_{11} &= Q_{12} = Q_{21} = Q_{22} = 0 \end{aligned} \right\} \quad (4.77)$$

Using (4.81) and (4.75) in (4.53) and (4.54) respectively

$$\left. \begin{aligned} T_{11}(x) &= 4p_0, T_{12}(x) = T_{21}(x) = T_{22}(x) = 0, \\ T_{31}(x) &= T_{32}(x) = T_{41}(x) = T_{42}(x) = 0 \end{aligned} \right\} \quad (4.78)$$

Then using (4.78) and (4.76) in (4.49)-(4.54) we get

$$\Delta_1(t) = 4p_0 \left(\int_b^c - \int_d^e \right) \frac{y\theta(y)dy}{y^2 - t^2} + t^2 L_1 + L_2 \quad (4.79)$$

$$\left. \begin{aligned} \Delta_2(t) &= t^2 M_1 + M_2 \\ \Delta_3(t) &= t^2 R_1 + R_2 \\ \Delta_4(t) &= t^2 N_1 + N_2 \end{aligned} \right\} \quad (4.80)$$

Now evaluating integrals in (4.83) we get,

$$\Delta_1(t) = 4p_0 \left\{ \begin{aligned} &s_1 + s_2 |t^2 - b^2| + s_3 |(t^2 - b^2)(c^2 - t^2)| \\ &+ s_4 |(t^2 - b^2)(c^2 - t^2)(d^2 - t^2)| + \frac{\pi}{2} \theta(t) \end{aligned} \right\} \quad (4.81)$$

$$\left. \begin{aligned} s_1 &= \left(\int_b^c - \int_d^e \right) y\theta_3(y)dy, s_2 = \left(\int_b^c - \int_d^e \right) y\theta_2(y)dy \\ s_3 &= \left(\int_b^c - \int_d^e \right) y\theta_1(y)dy, s_4 = \left(\int_b^c - \int_d^e \right) \frac{ydy}{\theta(y)} \\ \theta_1(y) &= (e^2 - y^2)^{1/2}, \theta_2(y) = \left[|(d^2 - y^2)| |\theta_1(y)| \right]^{1/2} \\ \theta_3 &= \left[|(c^2 - y^2)| |\theta_2(y)| \right]^{1/2} \end{aligned} \right\} \quad (4.82)$$

It is easy to compute the integrals shown above using mathematics. In sections (4.58), (4.60), (4.62), (4.64), and (4.69), we may now apply sections (4.75) through (4.76).

$$\left. \begin{aligned} U_1(x) = 0, F_1(x) = 0, F_2(x) = 0, F_3(x) = 0, F_4(x) = 0 \\ s_3 = \frac{1}{3}[(e^2 - c^2)^{3/2} - (e^2 - b^2)^{3/2} - (e^2 - d^2)^{3/2}] \end{aligned} \right\} \quad (4.83)$$

Now we can use the connection (4.44) - (4.47) in (4.52) and the equations (4.84) - (4.88) to get

$$g_{21}(t) = g_{22}(t) = g_{31}(t) = g_{32}(t) = g_{41}(t) = g_{42}(t) = 0 \quad (4.84)$$

$$\left. \begin{aligned} g_{11}(t) &= \frac{2p_0\Delta_5(t)}{\pi^2\theta(t)}, b < t < c \\ g_{12}(t) &= -\frac{2p_0\Delta_5(t)}{\pi^2\theta(t)}, d < t < e \\ \Delta_5(t) &= \Delta_1(t) + t^2L_1 + L_2 \end{aligned} \right\} \quad (4.85)$$

4.81 defines $\Delta(t)$. After using (4.85), we may proceed to evaluate the integrals in (4.57).

$$\sigma_{yy}^{s_3}(x, 0) = \frac{p_0}{\pi} \begin{cases} \frac{\Delta_5(x)}{\theta(x)}, x \in I_1 \\ -\frac{\Delta_5(x)}{\theta(x)}, x \in I_3 \\ \frac{\Delta_5(x)}{\theta(x)}, x \in I_5 \end{cases}$$

$$\sigma_{yy}^{a_3}(x, 0) = \sigma_{yy}^{a_4}(x, 0) = 0$$

$$\sigma_{xy}^{s_3}(x, 0) = \sigma_{xy}^{s_4}(x, 0) = \sigma_{xy}^{a_3}(x, 0) = \sigma_{xy}^{a_4}(x, 0) = 0$$

(4.72) says that the stress-intensity factor

$$\left. \begin{aligned} [K_b, K_c, K_d, K_e] &= \frac{p_0}{\pi} \left[\frac{\Delta_5(b)}{n_1(b)}, -\frac{\Delta_5(c)}{n_1(c)}, -\frac{\Delta_5(d)}{n_2(d)}, \frac{\Delta_5(e)}{n_2(e)} \right] \\ n_1(x) &= \sqrt{2x(c^2 - b^2)(d^2 - x^2)(e^2 - x^2)} \\ n_2(x) &= \sqrt{2x(e^2 - d^2)(x^2 - b^2)(x^2 - c^2)} \end{aligned} \right\} \quad (4.86)$$

With the use of these integrals, we can calculate the exact value of the displacement at the crack opening.

$$u_y^{s_3}(x, 0) = u_y^{s_4}(x, 0) = \frac{2(1 + \eta)}{\pi E} \left\{ \begin{aligned} &\int_x^c g_{11}(t) dt \\ &\int_x^d g_{12}(t) dt \end{aligned} \right\} \quad (4.87)$$

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$$\left. \begin{aligned} [p_1^\pm, p_2^\pm] &= p_0 [\delta(x - d_1), \delta(x + d_1)] \delta(y), b < d_1 < c \\ [p_3^\pm, p_4^\pm] &= p_0 [\delta(x - d_2), \delta(x + d_2)] \delta(y), d < d_2 < e \end{aligned} \right\} \quad (4.88)$$

$$q_i^\pm = 0, i = 1, 2, 3, 4$$

The issue is now limited to symmetric cases only.

By applying point forces of comparable size at certain locations, both internal and external fissures may burst $(\pm d_1, 0)$ and $(\pm d_2, 0)$ of crack faces.

$$\left. \begin{aligned} p_1^{e_1} &= p_1^{e_2} = p_1^{0_1} = p_1^{0_2} = 0, Q_1^{e_1} = Q_1^{e_2} = Q_1^{0_1} = Q_1^{0_2} = 0 \\ F_{ij} &= 0, \quad i = 1, 2, 3, 4; \quad i = 1, 2 \\ P_{11} &= 2p_0 [\delta(x - d_1) + \delta(x + d_1)], P_{12} = 2p_0 [\delta(x - d_2) + \delta(x + d_2)], \\ P_{21} &= P_{22} = Q_{11} = Q_{12} = Q_{21} = Q_{22} = 0 \end{aligned} \right\} \quad (4.89)$$

$$\int_b^c g_{i1}(t) dt = 0, \int_d^e g_{i2}(t) dt = 0, i = 1, 2, 3, 4 \quad (4.90)$$

For $i = 1, 2, 3, 4$, and $j = 1, 2$, we can get the value of $g_{ij}(t)$ using the equations (4.49) - (4.52) and (4.88) - (4.90).

$$g_{11}(t) = \frac{2p_0}{\theta(t)\pi^2} \left[G(t) + \left(\frac{d_1\theta(d_1)}{d_1^2 - t^2} - \frac{d_2\theta(d_2)}{d_2^2 - t^2} \right) \right] + t^2 M_1 + M_2 \quad (4.91)$$

$$M_1 = \frac{\alpha_7}{\alpha_8}, M_2 = \frac{\alpha_9}{\alpha_8} \quad (4.92)$$

$$\alpha_7 = \alpha_3\alpha_4 - \alpha_1\alpha_6, \alpha_8 = \alpha_2\alpha_6 - \alpha_3\alpha_5, \alpha_9 = \alpha_1\alpha_5 - \alpha_2\alpha_4 \quad (4.93)$$

$$\left. \begin{aligned} \alpha_1 &= \int_b^c G(t) dt \left\langle \frac{d_1\theta(d_1)}{d_1^2 - t^2} - \frac{d_2\theta(d_2)}{d_2^2 - t^2} \right\rangle \\ \alpha_2 &= \frac{c^3 - b^3}{3} \\ \alpha_3 &= (c - b) \end{aligned} \right\} \quad (4.94)$$

$$\alpha_4 = \int_d^e \frac{G(t) dt}{\theta(t)}, \alpha_5 = \frac{e^3 - d^3}{3}, \alpha_6 = e - d \quad (4.95)$$

$$g_{ij}(t) = 0, i = 2, 3, 4; j = 1, 2 \quad (4.96)$$

Now using $g_{11}(t)$, $g_{12}(t)$ utilizing the definitions in (4.71) and plugging the results into (4.57), we get,

$$\left. \begin{aligned} k_b &= \frac{p_0}{\pi} \frac{\Delta_5(b)}{n_1(b)} \\ k_c &= -\frac{p_0}{\pi} \frac{\Delta_5(c)}{n_1(c)} \\ k_d &= \frac{p_0}{\pi} \frac{\Delta_5(d)}{n_2(d)} \\ k_e &= \frac{p_0}{\pi} \frac{\Delta_5(e)}{n_2(e)} \end{aligned} \right\} \quad (4.97)$$

$$\left. \begin{aligned} n_1(x) &= [2x(c^2 - b^2)(e^2 - x^2)(d^2 - x^2)]^{1/2} \\ n_2(x) &= [2x(e^2 - d^2)(x^2 - b^2)(x^2 - e^2)]^{1/2} \end{aligned} \right\} \quad (4.98)$$

The numerical evaluation of crack shapes is based on the values for fracture opening displacement, which in turn determine the crack form.

$$u_y^{s_3}(x, 0) = u_y^{s_4}(x, 0) = \frac{2(1 - \eta^2)}{\pi E} \left\{ \begin{aligned} &\int_x^c g_{11}(t) dt \\ &\int_x^e g_{12}(t) dt \end{aligned} \right\} \quad (4.99)$$

4.2 FOUR ASYMMETRICALLY OPEN GRIFFITH CRACKS IN ORTHOPAEDIC INFINITE MEDIUM STRESS-INTENSITY FACTORS

4.2.1 Introduction

The current fad is for composites instead of materials made from natural sources. Bandopadhyaya S.N. and Murthy P.M. (1975) and Sharma and Sudhakar (1980) provided experimental and analytical evidence that supports the concept that composite materials are orthotropic continuum.

In this investigation, we sought to resolve an orthotropic multiply-connected infinite medium functioning under plane-strain circumstances. With four G-cracks present, the m-b value problem becomes much worse due to the mandated stresses at the crack face and displacement components throughout the whole boundary.

The cracks occupy the space in infinite orthotropic medium as $y=0$, $b < |x| < c$, $d < |x| < e < \infty$ with $b < c < d < e$. This m-b value issue is the

physical solution to the four Gcracks that were opened by asymmetrical pressures at the crack faces (figure 4.2).

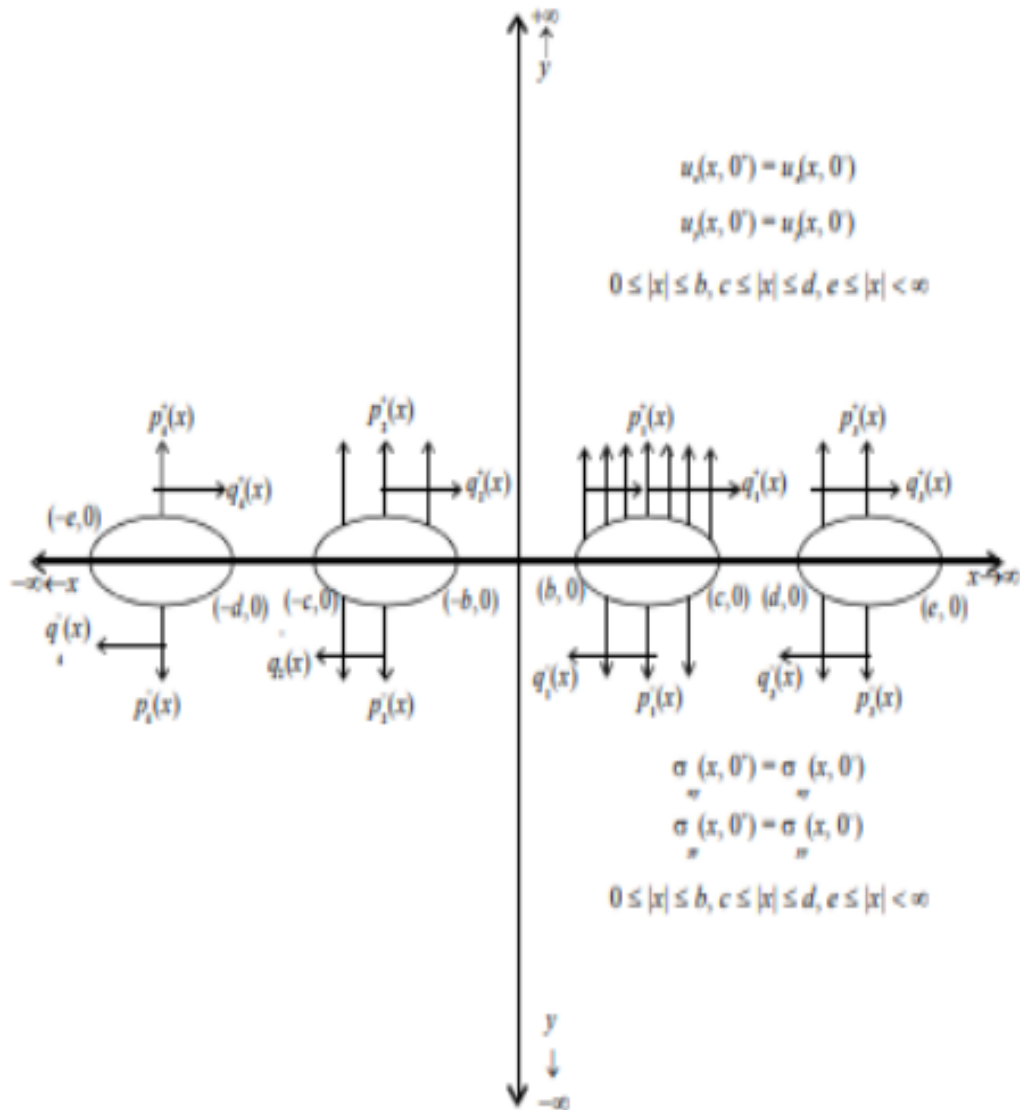


Figure 4.2: Asymmetrical forces at fracture faces in orthotropic infinite media cause crack opening geometry.

$$\sigma_{xy}(x, 0^\pm) = \begin{cases} q_1^\pm(x), & x \in I_1^\pm \\ q_2^\pm(x), & x \in I_2^\pm \\ q_3^\pm(x), & x \in I_3^\pm \\ q_4^\pm(x), & x \in I_4^\pm \end{cases} \quad (4.100)$$

$$\sigma_{yy}(x, 0^\pm) = \begin{cases} p_1^\pm(x), & x \in I_1^\pm \\ p_2^\pm(x), & x \in I_2^\pm \\ p_3^\pm(x), & x \in I_3^\pm \\ p_4^\pm(x), & x \in I_4^\pm \end{cases} \quad (4.101)$$

$$\text{where, } I_1^\pm = (b, c), I_2^\pm = (-c, -b), I_3^\pm = (d, e), I_4^\pm = (-e, -d) \quad (4.102)$$

At $y=0$, all boundaries except the crack faces satisfy the continuity criteria, and $(\pm)$ over amounts denote $y > 0$ and $y < 0$, respectively.

$$u_x(x, 0^+) = u_x(x, 0^-), \quad u_y(x, 0^+) = u_y(x, 0^-) \quad (4.103)$$

$$\sigma_{xy}(x, 0^+) = \sigma_{xy}(x, 0^-), \quad \sigma_{yy}(x, 0^+) = \sigma_{yy}(x, 0^-) \quad (4.104)$$

for $0 \leq |x| \leq b, \quad c \leq |x| \leq d, \quad e \leq |x| < \infty$. (u_x, u_y) and $(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$ are

Stress and displacement components at a broad medium point (x, y) .

The assumption is also that all physical equations become zero as $\sqrt{x^2 + y^2} \rightarrow \infty$. We checked throughout that, see Burniston [45].

$$[u_y(x, 0^+), u_y(x_1, 0^-)] > 0, x \in I_1^\pm \cup I_2^\pm \cup I_3^\pm \cup I_4^\pm \quad (4.105)$$

Thus it's clear that spaces are really opening up. Here is the definition of the Fourier transform:

With usual inversion formula. (Burniston E.E.,1969)

4.2.2 Formulation

Using the approach of (Kushwaha, 1978), we are able to solve the current linear elasticity issue. At each location in the medium, we separate the physical quantities into symm and anti-symm quantities as

$$\sigma_{ij}(x, y) = \sigma_{ij}^{s_k}(x, y) + \sigma_{ij}^{a_k}(x, y), i, j = x, y \quad (4.106)$$

$$u_i(x, y) = u_i^{s_k}(x, y) + u_i^{a_k}(x, y), \quad i = x, y \quad (4.107)$$

$y > 0$ and $y < 0$ are denoted by $k = 3, 4$, respectively.

"S k" and "a k" stand for the symm and anti-symm parts of the problem.

Originally a two-dimensional space, $(-\infty, \infty) \cup (-\infty, \infty)$ i.e., the solution space is now $[0, \infty) \cup (-\infty, \infty)$ i.e. $x \geq 0, y = 0$. Now, we assume the displacement components as

$$u_x^{s_k}(x, y) = \int_0^\infty \frac{\sin(\xi x)}{\xi} \left[a_{11} \frac{\partial^2 G^{s_k}}{\partial y^2} - a_{12} \xi^2 G^{s_k} \right] d\xi \quad (4.108)$$

$$u_x^{a_k}(x, y) = \int_0^\infty \frac{\cos(\xi x)}{\xi} \left[a_{11} \frac{\partial^2 G^{a_k}}{\partial y^2} - a_{12} \xi^2 G^{a_k} \right] d\xi \quad (4.109)$$

$$u_y^{s_k}(x, y) = \int_0^\infty \frac{\cos(\xi x)}{\xi^2} \left[a_{11} \frac{\partial^3 G^{s_k}}{\partial y^3} - (2a_{12} + a_{66}) \xi^2 \frac{\partial G^{s_k}}{\partial y} \right] d\xi \quad (4.110)$$

$$u_y^{a_k}(x, y) = \int_0^\infty \frac{\sin(\xi x)}{\xi^2} \left[a_{11} \frac{\partial^3 G^{a_k}}{\partial y^3} - (2a_{12} + a_{66}) \xi^2 \frac{\partial G^{a_k}}{\partial y} \right] d\xi \quad (4.111)$$

$$\left. \begin{aligned} (r_1 - r_2) G^{S_k}(\xi, y) &= [(r_1 - r_2) A_{k-2}(\xi) - B_{k-2}(\xi)] e^{\mp r_1 \xi y} + B_{k-2}(\xi) e^{\mp r_2 \xi y} \\ (r_1 - r_2) G^{a_k}(\xi, y) &= [(r_1 - r_2) C_{k-2}(\xi) - D_{k-2}(\xi)] e^{\mp r_1 \xi y} + D_{k-2}(\xi) e^{\mp r_2 \xi y} \end{aligned} \right\} \quad (4.112)$$

Where r_1^2 and r_2^2 are the roots of

$$r^4 - 2B_1 r^2 + B_2 = 0, \quad B_1 = \frac{2a_{12} + a_{66}}{a_{11}}, \quad B_2 = a_{22} / a_{11}$$

And $a_{11} \sim a_{66}$ are elastic constants of the medium.

In case of $k = 3, 4$ and 0 respectively, the $(-)$ and $(+)$ are to be used that is, y is greater than 0 or less than 0 respectively. Besides the equation of equilibrium, the above assumptions concerning the displacement components also provide compatibility relations and continuity relations.

If eight random constants in (4.112) can be found, the issue can be solved mathematically. Using the S-S relations and the relations (4.108) through (4.112)

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \quad (4.113)$$

$$\sigma_{xy}^{S_k}(x, y) = \pm \int_0^\infty \xi^2 \frac{\sin(\xi x)}{(r_1 - r_2)} \{ [(r_1 - r_2) A_{k-2} - B_{k-2}] e^{\mp r_1 \xi y} + B_{k-2} e^{\mp r_2 \xi y} \} d\xi \quad (4.114)$$

$$\sigma_{xy}^{a_k}(x, y) = \pm \int_0^\infty \xi^2 \frac{\cos(\xi x)}{(r_1 - r_2)} \{ [(r_1 - r_2) C_{k-2} - D_{k-2}] e^{\mp r_1 \xi y} + D_{k-2} e^{\mp r_2 \xi y} \} d\xi \quad (4.115)$$

$$\sigma_{yy}^{S_k}(x, y) = - \int_0^\infty \xi^2 \frac{\cos(\xi x)}{(r_1 - r_2)} \{ [(r_1 - r_2) A_{k-2} - B_{k-2}] e^{\mp r_1 \xi y} + B_{k-2} e^{\mp r_2 \xi y} \} d\xi \quad (4.116)$$

$$\sigma_{yy}^{a_k}(x, y) = -\int_0^\infty \xi^2 \frac{\sin(\xi x)}{(r_1 - r_2)} \{[(r_1 - r_2)C_{k-2} - D_{k-2}]e^{\mp r_1 \xi y} + D_{k-2}e^{\mp r_2 \xi y}\} d\xi \quad (4.117)$$

$$\sigma_{xx}^{s_k}(x, y) = \int_0^\infty \xi^2 \frac{\cos(\xi x)}{r_1 - r_2} \{[(r_1 - r_2)A_{k-2} - B_{k-2}]r_1^2 e^{\mp r_1 \xi y} + r_2^2 B_{k-2}e^{\mp r_2 \xi y}\} d\xi \quad (4.118)$$

$$\sigma_{xx}^{a_k}(x, y) = \int_0^\infty \xi^2 \frac{\cos(\xi x)}{r_1 - r_2} \{[(r_1 - r_2)C_{k-2} - D_{k-2}]r_1^2 e^{\mp r_1 \xi y} + D_{k-2}r_2^2 e^{\mp r_2 \xi y}\} d\xi \quad (4.119)$$

4.2.3 Equation For Integral Quintuple Reductions

In order for physical quantities to be considered continuous, the continuity criteria must be met for both symm and anti-symm independently. With the boundary conditions (4.100)–(4.101) and the continuity criteria (4.104), we may invert the Fourier integral transform and get,

$$A_1 - A_2 = \xi^{-2} \left[\left\langle \int_b^c p_1^{e_1}(x) + \int_d^e p_1^{e_2}(x) \right\rangle \cos(\xi x) dx \right] \quad (4.120)$$

$$C_1 - C_2 = \xi^{-2} \left[\left\langle \int_b^c p_1^{0_1}(x) + \int_d^e p_1^{0_2}(x) \right\rangle \sin(\xi x) dx \right] \quad (4.121)$$

$$\xi(A_1 + A_2) - (B_1 - B_2) = \xi^{-1} \left[\left\langle \int_b^c Q_1^{e_1}(x) + \int_d^e Q_1^{e_2}(x) \right\rangle \sin(\xi x) dx \right] \quad (4.122)$$

$$\xi(C_1 + C_2) - (D_1 - D_2) = \xi^{-1} \left[\left\langle \int_b^c Q_1^{0_1}(x) + \int_d^e Q_1^{0_2}(x) \right\rangle \cos(\xi x) dx \right] \quad (4.123)$$

Where

$$\left. \begin{aligned}
 p_1^{e_1}(x) &= p_1^+ + p_2^+ - (p_1^- + p_2^-) \\
 p_1^{e_2}(x) &= p_3^+ + p_4^+ - (p_3^- + p_4^-) \\
 p_1^{0_1}(x) &= p_1^+ - p_2^+ - (p_1^- - p_2^-) \\
 p_1^{0_2}(x) &= p_3^+ - p_4^+ - (p_3^- - p_4^-) \\
 Q_1^{e_1}(x) &= q_1^- - q_2^- - (q_1^+ - q_2^+) \\
 Q_1^{e_2}(x) &= q_3^- - q_4^- - (q_3^+ - q_4^+) \\
 Q_1^{0_1}(x) &= q_1^- + q_2^- - (q_1^+ + q_2^+) \\
 Q_1^{0_2}(x) &= q_3^- + q_4^- - (q_3^+ + q_4^+)
 \end{aligned} \right\} \quad (4.124)$$

Equations (4.120) through (4.123) are obtained, allowing us to find four constants. According to these m-b criteria, the other four constants will be found. We now divide the system into symm and anti-symm parts by using the continuity criteria of the displacement components provided by (4.103). After using (4.120) - (4.124), we get the following by utilizing (4.103) and (4.108) - (4.115):

$$\int_0^\infty \xi(B_1 - B_2) \cos(\xi x) d\xi = t_1 \begin{cases} F_{11}(x), & x \in I_1 \\ F_{12}(x), & x \in I_3 \\ F_{13}(x), & x \in I_5 \end{cases} \quad (4.125)$$

$$\int_0^\infty \xi(D_1 - D_2) \sin(\xi x) d\xi = t_1 \begin{cases} F_{21}(x), & x \in I_1 \\ F_{22}(x), & x \in I_3 \\ F_{23}(x), & x \in I_5 \end{cases} \quad (4.126)$$

$$\int_0^\infty \xi(B_1 + B_2) \sin(\xi x) d\xi = t_2 \begin{cases} F_{31}(x), & x \in I_1 \\ F_{32}(x), & x \in I_3 \\ F_{33}(x), & x \in I_5 \end{cases} \quad (4.127)$$

$$\int_0^\infty \xi(D_1 + D_2) \cos(\xi x) d\xi = t_2 \begin{cases} F_{41}(x), & x \in I_1 \\ F_{42}(x), & x \in I_3 \\ F_{43}(x), & x \in I_5 \end{cases} \quad (4.128)$$

$$\left. \begin{aligned} \alpha_3 &= \frac{\alpha_1}{r_1} - \alpha_2 / (r_1 - r_2), \alpha_i = \frac{a_{11}r_i^2 - a_{12}}{r_1 - r_2} \\ F_{13} &= F_{21} = F_{31} = F_{43} = 0, \\ I_1 &= [0, b], I_3 = [c, d], I_5 = [e, \infty) \\ t_1 &= -\frac{1}{r_1\alpha_3}, t_2 = \frac{\alpha_3}{r_1 + r_2}, \end{aligned} \right\} \quad (4.129)$$

$$\left. \begin{aligned} F_{11} &= \frac{\pi}{2} \left\langle \int_b^c Q_1^{e_1}(x) dx + \int_d^e Q_1^{e_2}(x) dx \right\rangle, F_{12} = \frac{\pi}{2} \int_d^e Q_1^{e_2}(x) dx, \\ F_{22} &= \frac{\pi}{2} \int_b^c Q_1^{0_1}(x) dx, F_{23} = \frac{\pi}{2} \int_d^e Q_1^{0_2}(x) dx, \\ F_{32} &= \frac{\pi}{2} \int_b^c p_1^{e_1}(x) dx, F_{33} = \frac{\pi}{2} \int_d^e p_1^{e_2}(x) dx + F_{32}, \\ F_{41} &= \frac{\pi}{2} \int_b^c p_1^{0_1}(x) dx + F_{42}, F_{42} = \frac{\pi}{2} \int_d^e p_1^{0_2}(x) dx \end{aligned} \right\} \quad (4.130)$$

Now we evaluate $(\sigma_{yy}^{(3)} + \sigma_{yy}^{(4)})$ and $(\sigma_{xy}^{(3)} + \sigma_{xy}^{(4)})$ for symm & anti-symm by using (4.114) – (4.119) with (4.120) – (4.123) we get,

$$\int_0^\infty \xi^2 (B_1 - B_2) \cos(\xi x) d\xi = \begin{cases} P_{11} + \int_b^c \frac{y Q_1^{e_1}(y) dy}{y^2 - x^2}, x \in I_2 \\ P_{12} + \int_b^e \frac{y Q_1^{e_2}(y) dy}{y^2 - x^2}, x \in I_4 \end{cases} \quad (4.131)$$

$$\int_0^\infty \xi^2 (D_1 - D_2) \sin(\xi x) d\xi = \begin{cases} P_{21} + x \int_b^c \frac{Q_1^{0_1}(y) dy}{y^2 - x^2}, x \in I_2 \\ P_{22} + x \int_b^e \frac{Q_1^{0_2}(y) dy}{y^2 - x^2}, x \in I_4 \end{cases} \quad (4.132)$$

$$\int_0^\infty \xi^2 (B_1 + B_2) \sin(\xi x) d\xi = \begin{cases} Q_{11} + x \int_b^c \frac{P_1^{e_1}(y) dy}{y^2 - x^2}, x \in I_2 \\ Q_{12} + x \int_d^e \frac{P_1^{e_2}(y) dy}{y^2 - x^2}, x \in I_4 \end{cases} \quad (4.133)$$

$$\int_0^\infty \xi^2 (D_1 + D_2) \cos(\xi x) d\xi = \begin{cases} Q_{21} + \int_b^c \frac{P_1^{0_1}(y) dy}{y^2 - x^2}, x \in I_2 \\ Q_{22} + \int_d^e \frac{P_1^{0_2}(y) dy}{y^2 - x^2}, x \in I_4 \end{cases} \quad (4.134)$$

Where

$$\left. \begin{aligned} I_2 &= (b, c), I_4 = (d, e) \text{ and} \\ P_{11} &= p_1^+ + p_1^- + p_2^+ + p_2^-, \\ P_{12} &= p_3^+ + p_3^- + p_4^+ + p_4^-, \\ P_{21} &= p_1^+ - p_1^- + (p_2^+ - p_2^-), \\ P_{22} &= p_3^+ - p_3^- + (p_4^+ - p_4^-), \\ Q_{11} &= q_1^+ + q_1^- + q_2^+ + q_2^-, \\ Q_{12} &= q_3^+ + q_3^- + q_4^+ + q_4^-, \\ Q_{21} &= q_1^+ - q_1^- + q_2^+ - q_2^-, \\ Q_{22} &= q_3^+ - q_3^- + q_4^+ - q_4^- \end{aligned} \right\} \quad (4.135)$$

Thus remaining four constants $B_1, B_2; D_1, D_2$ found by solving the quintuple integral equations (IE) provided by (4.125) - (4.129) and (4.131) - (4.134).

4.2.4 Quintuple integral equations (IE) solved

See (Kushwaha P.S., 1975) for the hypothesised trial solutions to the aforementioned quintuple integral equations.

$$B_1 - B_2 = \frac{2}{\pi\xi} \left[\left\langle \int_b^c g_{11}(t) + \int_d^e g_{12}(t) \right\rangle \sin(\xi t) dt \right. \\ \left. - t_1 \left\langle \int_c^d F'_{11}(t) + \int_e^\infty F'_{12}(t) \right\rangle \sin(\xi t) dt \right] \quad (4.136)$$

$$D_1 - D_2 = \frac{2}{\pi\xi} \left[\left\langle \int_b^c g_{21}(t) + \int_d^e g_{22}(t) \right\rangle \right. \\ \left. - t_1 \left\langle \int_c^d F'_{22}(t) + \int_e^\infty F'_{23}(t) \right\rangle \right] (1 - \cos \xi t) dt \quad (4.137)$$

$$B_1 + B_2 = \frac{2}{\pi\xi} \left[\left\langle \int_b^c g_{31}(t) + \int_d^e g_{32}(t) \right\rangle \right. \\ \left. - t_1 \left\langle \int_c^d F'_{32}(t) + \int_e^\infty F'_{33}(t) \right\rangle \right] (1 - \cos \xi t) dt \quad (4.138)$$

$$D_1 + D_2 = \frac{2}{\pi\xi} \left[\left\langle \int_b^c g_{41}(t) + \int_d^e g_{42}(t) \right\rangle \right. \\ \left. - t_1 \left\langle \int_c^d F'_{41}(t) + \int_e^\infty F'_{42}(t) \right\rangle \right] \sin(\xi t) dt \quad (4.139)$$

The above assumption (4.136) – (4.139) satisfies the equations (4.125) – (4.128) identically if

$$\left. \begin{aligned} \int_b^c g_{11}(t) dt &= t_1 \langle F_{12}(d) - F_{11}(b) \rangle \\ \int_d^e g_{12}(t) dt &= t_1 F_{12}(d) \end{aligned} \right\} \quad (4.140)$$

$$\left. \begin{aligned} \int_b^c g_{21}(t) dt &= -t_1 F_{12}(d) \\ \int_d^e g_{22}(t) dt &= t_1 \langle F_{22}(d) - F_{23}(e) \rangle \end{aligned} \right\} \quad (4.141)$$

$$\left. \begin{aligned} \int_b^c g_{31}(t) dt &= -t_2 F_{23}(e) \\ \int_d^e g_{32}(t) dt &= t_2 \langle F_{32}(d) - F_{33}(e) \rangle \end{aligned} \right\} \quad (4.142)$$

$$\left. \begin{aligned} \int_b^c g_{41}(t) dt &= t_2 \langle F_{42}(c) - F_{41}(b) \rangle \\ \int_d^e g_{42}(t) dt &= -t_2 F_{42}(d) \end{aligned} \right\} \quad (4.143)$$

Where F_{ij} are given in (4.133). Now, the substitution of (4.136) – (4.139) into (4.131) – (4.134) and using the method of (Kushwaha P.S.,1975) for inverting the integrals we get,

$$\left. \begin{aligned} g_{11}(t) &= \frac{\Delta_1(t)}{\pi^2 \theta(t)}, \quad t \in I_2, \\ g_{12}(t) &= -\frac{\Delta_1(t)}{\pi^2 \theta(t)}, \quad t \in I_4 \end{aligned} \right\} \quad (4.144)$$

$$\left. \begin{aligned} g_{21}(t) &= \frac{t\Delta_2(t)}{\pi^2 \theta(t)}, \quad t \in I_2, \\ g_{22}(t) &= -\frac{t\Delta_2(t)}{\pi^2 \theta(t)}, \quad t \in I_4 \end{aligned} \right\} \quad (4.145)$$

$$\left. \begin{aligned} g_{31}(t) &= \frac{t\Delta_3(t)}{\pi^2 \theta(t)}, \quad t \in I_2, \\ g_{32}(t) &= -\frac{t\Delta_3(t)}{\pi^2 \theta(t)}, \quad t \in I_4 \end{aligned} \right\} \quad (4.146)$$

$$\left. \begin{aligned} g_{41}(t) &= \frac{\Delta_4(t)}{\pi^2 \theta(t)}, \quad t \in I_2, \\ g_{42}(t) &= -\frac{\Delta_4(t)}{\pi^2 \theta(t)}, \quad t \in I_4 \end{aligned} \right\} \quad (4.147)$$

$$\text{with } \theta(t) = \left\{ (t^2 - b^2)(c^2 - t^2)(d^2 - t^2)(e^2 - t^2) \right\}^{1/2} \quad (4.148)$$

$$\Delta_1(t) = \left\langle \int_b^c T_{11} - \int_d^e T_{12} \right\rangle \frac{y\theta(y)dy}{y^2 - t^2} + t_1 \left\langle \int_c^d F'_{11} \right\rangle \frac{y\theta(y)}{y^2 - t^2} dy + t^2 L_1 + L_2 \quad (4.149)$$

$$\Delta_2(t) = \left\langle \int_b^c T_{21} - \int_d^e T_{22} \right\rangle \frac{\theta(y)dy}{y^2 - t^2} + t_1 \left\langle \int_c^d F'_{22} - \int_e^\infty F'_{23} \right\rangle \frac{\theta(y)dy}{y^2 - t^2} + t^2 M_1 + M_2 \quad (4.150)$$

$$\Delta_3(t) = \left\langle \int_b^c T_{31} - \int_d^e T_{32} \right\rangle \frac{\theta(y)dy}{y^2 - t^2} + t_1 \left\langle \int_c^d F'_{32} - \int_e^\infty F'_{33} \right\rangle \frac{\theta(y)dy}{y^2 - t^2} + t^2 R_1 + R_2 \quad (4.151)$$

$$\Delta_4(t) = \left\langle \int_b^c T_{41} - \int_d^e T_{42} \right\rangle \frac{\theta(y)dy}{y^2 - t^2} + t_1 \left\langle \int_c^d F'_{41} \right\rangle \frac{\theta(y)dy}{y^2 - t^2} + t^2 N_1 + N_2 \quad (4.152)$$

Where $L_i, M_i, N_i, R_i, i=1,2$ are constants to be determined through (4.144) – (4.147) and (4.140) – (4.143). And,

$$\left. \begin{aligned} T_{11}(x) &= P_{11}(x) - \int_b^c \frac{y Q_1^e(y) dy}{y^2 - x^2} \\ T_{12} &= P_{12}(x) - \int_d^e \frac{y Q_1^e(y) dy}{y^2 - x^2} \end{aligned} \right\} \quad (4.153)$$

$$\left. \begin{aligned} T_{21}(x) &= P_{21}(x) - \int_b^c \frac{x Q_1^{0_1} dy}{y^2 - x^2} \\ T_{22}(x) &= P_{12}(x) - \int_d^e \frac{x Q_1^{0_2} dy}{y^2 - x^2} \end{aligned} \right\} \quad (4.154)$$

$$\left. \begin{aligned} T_{31}(x) &= Q_{11}(x) + x \int_b^c \frac{P_1^{e_1} dy}{y^2 - x^2} \\ T_{32}(x) &= Q_{12}(x) + x \int_d^e \frac{P_1^{e_2} dy}{y^2 - x^2} \end{aligned} \right\} \quad (4.155)$$

$$\left. \begin{aligned} T_{41}(x) &= Q_{21}(x) + \int_b^c \frac{y P_1^{0_1} dy}{y^2 - x^2} \\ T_{42}(x) &= Q_{22}(x) + \int_d^e \frac{y P_1^{0_2} dy}{y^2 - x^2} \end{aligned} \right\} \quad (4.156)$$

Evaluation of physical quantities in terms of solutions to integral equations (IE) will be covered in the following section.

4.2.5 Physical Quantities

S.I.F. at crack tips and the c.o.d. are the physical parameters that are practically important in fracture mechanics.

Stress-Components

The stress components are evaluated for $y = 0$ $x \in [0, b] \cup [c, d] \cup [e, \infty)$. The

normal component of stress is $\sigma_{yy}(x, 0)$ and shear stress is $\sigma_{xy}(x, 0)$.

Normal Stress

The normal stress is evaluated through the value of integrals (4.116) -

for symm & anti-symm parts, respectively for $x \in I_2 \cup I_3 \cup I_5$.

$$\sigma_{yy}^{s_3}(x, 0) = \sigma_{yy}^{s_4}(x, 0), \quad x \in I_1 \cup I_3 \cup I_5$$

$$\sigma_{yy}^{s_3}(x, 0) = \frac{2}{\pi} \left[\left\{ \int_b^c g_{11}(t) + \int_d^e g_{12}(t) - t_1 \left\langle \int_c^d F'_{12}(t) \right\rangle \right\} \frac{t dt}{t^2 - x^2} + \frac{\pi}{2} F_1(x) \right],$$

with (4.157)

$$F_1(x) = \begin{cases} \int_b^c Q_1^{e_1}(x) dx + \int_d^e Q_1^{e_2}(x) dx, & x \in I_1 \\ \int_d^e Q_1^{e_2}(x) dx, & x \in I_3 \\ 0, & x \in I_5 \end{cases} \quad (4.158)$$

$$\left. \begin{aligned} \sigma_{yy}^{a_3}(x, 0) &= \sigma_{yy}^{a_4}(x, 0) \\ &= \frac{1}{2\pi} \left[\int_b^c g_{21}(t) + \int_d^e g_{22}(t) - t_1 \left\langle \int_c^d F'_{22}(t) + \int_e^\infty F'_{23}(t) \right\rangle \right] \frac{x dt}{t^2 - x^2} + \frac{\pi}{2} F_2(x) \end{aligned} \right\} \quad (4.159)$$

$$F_2(x) = \begin{cases} \int_b^c Q_1^{0_1} dt + \int_d^t Q_1^{0_2} dt, & x \in I_1 \\ \int_d^e Q_1^{0_2}(t) dt, & x \in I_3 \\ 0, & x \in I_5 \end{cases} \quad (4.160)$$

Shear Stress

The component of shear $\sigma_{xy}(x, 0^+) = \sigma_{xy}(x, 0^-)$ for $x \in I_1 \cup I_3 \cup I_5$ is assessed by calculating the value of the integrals on the left side of equations (4.133) through (4.1384), as follows:

$$\left. \begin{aligned} \sigma_{xy}^{s_3}(x, 0) &= \sigma_{xy}^{s_4}(x, 0) \\ &= \frac{x}{2\pi} \left[\left\{ \int_b^c g_{31}(t) + \int_d^e g_{32}(t) + t_1 \left\langle \int_c^d F'_{32}(t) + \int_c^\infty F'_{33}(t) \right\rangle \right\} \frac{dt}{t^2 - x^2} \right] \\ &\quad + \frac{\pi}{2} F_3(x), \quad x \in I_1 \cup I_3 \cup I_5 \end{aligned} \right\} \quad (4.161)$$

$$F_3(x) = \begin{cases} 0, & x \in I_1 \\ \int_b^c P_1^{e_1}(y) dy, & x \in I_3 \\ \int_b^c P_1^{e_1}(y) dy + \int_d^e P_1^{e_2} dy, & x \in I_5 \end{cases}$$

And

$$\left. \begin{aligned} \sigma_{xy}^{a_3}(x, 0) &= -\frac{1}{2\pi} \left[\left\{ \int_b^c g_{41}(t) + \int_{g_{42}}^e (t) dt_1 \right. \right. \\ &\quad \left. \left. + t \left\langle \int_c^d F'_{42}(t) + \int_e^\infty F'_{43}(t) \right\rangle \right\} \frac{tdt}{t^2 - x^2} \right] + \frac{\pi}{2} F_4(x) \\ \sigma_{yy}^{a_3}(x, 0) &= \sigma_{yy}^{a_4}(x, 0) \end{aligned} \right\} \quad (4.163)$$

$$F_4(x) = \begin{cases} 0, x \in I_1 \\ \int_b^c P_1^{0_1}(y) dy, x \in I_3 \\ \int_b^c P_1^{0_1}(y) dy + \int_d^e P_1^{0_2}(y) dy, x \in I_5 \end{cases} \quad (4.164)$$

With $i = 1, 2, 3, 4$ and $j = 1, 2$ we get the following by computing the integrals and subtracting (4.144) from (4.147).

$$\sigma_{yy}^{s_3}(x, 0) = \begin{cases} \frac{\Delta_1(x)}{2\pi\theta(x)} + F_1(x), x \in I_1 \\ -\frac{\Delta_1(x)}{2\pi\theta(x)} + F_1(x), x \in I_3 \\ \frac{\Delta_1(x)}{2\pi\theta(x)} + F_1(x), x \in I_5 \end{cases} \quad (4.165)$$

Where $\theta(x)$ and $\Delta_1(x)$ are given by (4.148) and (4.149) and

$$\sigma_{yy}^{a_3}(x, 0) = \begin{cases} \frac{x}{2\pi} \frac{\Delta_2(x)}{\theta(x)} + \frac{\pi}{4} F_2(x), x \in I_1 \\ -\frac{x}{2\pi} \frac{\Delta_2(x)}{\theta(x)} + \frac{\pi}{4} F_2(x), x \in I_3 \\ \frac{x}{2\pi} \frac{\Delta_2(x)}{\theta(x)} + \frac{\pi}{4} F_2(x), x \in I_5 \end{cases} \quad (4.166)$$

Where $\Delta_2(x)F_2$ are given by (4.150) & (4.160), respectively

$$\sigma_{xy}^{s_3}(x, 0) = \begin{cases} \frac{x}{2} \frac{\Delta_3(x)}{\theta(x)} + \frac{\pi}{2} F_3(x), x \in I_1 \\ -\frac{x}{2} \frac{\Delta_3(x)}{\theta(x)} + \frac{\pi}{4} F_3(x), x \in I_3 \\ \frac{x}{2} \frac{\Delta_3(x)}{\theta(x)} + \frac{\pi}{4} F_3(x), x \in I_5 \end{cases} \quad (4.167)$$

where $\Delta_3(x)$ and $F_3(x)$ are given by (4.151) and (4.159), respectively.

$$\sigma_{xy}^{a_4}(x,0) = \begin{cases} \frac{\Delta_4(x)}{2\pi\theta(x)} + F_4(x), x \in I_1 \\ -\frac{\Delta_4(x)}{2\pi\theta(x)} + F_4(x), x \in I_3 \\ \frac{\Delta_4(x)}{2\pi\theta(x)} + F_4(x), x \in I_5 \end{cases} \quad (4.168)$$

Equations (4.152) and (4.164) provide $\Delta_4(x)$ and $F_4(x)$, respectively.

Crack Opening Displacement

We can learn the fracture's shape from the c.o.d. Critical operating differential (c.o.d.) can be calculated by applying the relationships (4.136) to (4.139), and integrals of the left-hand side of equations (4.128) to (4.129).

$$u_y^{s_3}(x,0) - u_y^{s_4}(x,0) = r_1 \alpha_3 \begin{cases} \int_x^c g_{11}(t) + t_1 F_{12}(d) - U_1(x), x \in I_2 \\ \int_x^e g_{12}(t) dt - U_1(x), x \in I_4 \end{cases} \quad (4.169)$$

$$U_1(x) = \frac{\pi}{4} \left[\int_b^c p_1^{e_1}(t) + \int_a^e p_1^{e_2}(t) \right] \log |t^2 - x^2| dt \quad (4.170)$$

$$u_y^{s_3}(x,0) + u_y^{s_4}(x,0) = r_1 \alpha_3 \begin{cases} \int_x^c g_{11}(t) - t_1 g_{31}(x) - t_1 (F_{12}(d) - F_{12}(c)), x \in I_2 \\ \int_x^e g_{12}(t) dt - t_1 g_{32}^{(x)}, x \in I_4 \end{cases} \quad (4.171)$$

Thus using (4.169) – (4.171) we can easily evaluate $u_y^{s_3}$ and $u_y^{s_4}$.

Stress-Intensity Factors

The S.I.F. is as a consequence of crack tips.

$$\left. \begin{aligned} [K_b, N_b] &= \lim_{x \rightarrow b^-} \sqrt{b-x} [\sigma_{yy}, \sigma_{xy}] \\ [K_c, N_c] &= \lim_{x \rightarrow c^+} \sqrt{x-c} [\sigma_{yy}, \sigma_{xy}] \\ [K_d, N_d] &= \lim_{x \rightarrow d^-} \sqrt{d-x} [\sigma_{yy}, \sigma_{xy}] \\ [K_e, N_e] &= \lim_{x \rightarrow e^+} \sqrt{x-e} [\sigma_{yy}, \sigma_{xy}] \end{aligned} \right\} \quad (4.172)$$

4.2.6 Special Type of Loading

In order to make the analysis and integral equation (IE) method as realistic as possible, we take into account specific loads sub-across fracture faces.

$$p_1^\pm = p_2^\pm = p_3^\pm = p_4^\pm = p_0 \quad (4.173)$$

$$q_1^\pm = q_2^\pm = q_3^\pm = q_4^\pm = 0 \quad (4.174)$$

In other words, the force acting uniformly and consistently across the fracture faces is causing the fissures to open. By use the formula (4.173)

– (4.174) in (4.124) – (4.127) we get

$$p_1^{\varepsilon_1}(x) = 0, p_1^{\varepsilon_2}(x) = 0, p_1^{0_1}(x) = 0, p_1^{0_2}(x) = 0 \quad (4.175)$$

$$Q_1^{\varepsilon_1}(x) = 0 = Q_1^{\varepsilon_2}(x) = Q_1^{0_1}(x) = Q_1^{0_2}(x) \quad (4.176)$$

Now we use (4.175) (4.176) in (4.133) we get,

$$\left. \begin{aligned} F_{11}(x) &= F_{12}(x) = F_{13}(x) = 0, \\ F_{21}(x) &= F_{22}(x) = F_{23}(x) = 0 \end{aligned} \right\} \quad (4.177)$$

$$\left. \begin{aligned} F_{31}(x) = F_{32}(x) = F_{33}(x) = 0, \\ F_{41}(x) = F_{42}(x) = F_{43}(x) = 0 \end{aligned} \right\} \quad (4.178)$$

Now using (4.175)-(4.176) we get

$$P_{11} = 4p_0 \quad P_{12} = P_{21} = P_{22} = 0 \quad (4.179)$$

$$Q_{11} = Q_{12} = Q_{21} = Q_{22} = 0 \quad (4.180)$$

Using (4.179)(4.180) and (4.175)-(4.176) in (4.153) - (4.154)

$$T_{11}(x) = 4p_0, T_{12}(x) = T_{21}(x) = T_{22}(x) = 0, \quad (4.181)$$

$$T_{31}(x) = T_{32}(x) = T_{41}(x) = T_{42}(x) = 0 \quad (4.182)$$

Then using (4.181) and (4.177) (4.178) in (4.149)-(4.152) we get,

$$\Delta_1(t) = 4p_0 \left(\int_b^c - \int_d^e \right) \frac{y\theta(y)dy}{y^2 - t^2} + t^2 L_1 + L_2, \quad (4.183)$$

$$\Delta_2(t) = t^2 M_1 + M_2 \quad (4.184)$$

$$\Delta_3(t) = t^2 R_1 + R_2 \quad (4.185)$$

$$\Delta_4(t) = t^2 N_1 + N_2 \quad (4.186)$$

Now evaluating integrals in (4.183) we get,

$$\Delta_1(t) = 4p_0 \left\{ \begin{aligned} &s_1 + s_2 |t^2 - b^2| + s_3 |(t^2 - b^2)(c^2 - t^2)| \\ &+ s_4 |(t^2 - b^2)(c^2 - t^2)(d^2 - t^2)| + \frac{\pi}{2} \theta(t) \end{aligned} \right\} \quad (4.187)$$

$$\left. \begin{aligned} s_1 &= \left(\int_b^c - \int_d^e \right) y \theta_3(y) dy, s_2 = \left(\int_b^c - \int_d^e \right) y \theta_2(y) dy \\ s_3 &= \left(\int_b^c - \int_d^e \right) y \theta_1(y) dy, s_4 = \left(\int_b^c - \int_d^e \right) \frac{y dy}{\theta(y)} \\ \theta_1(y) &= (e^2 - y^2)^{1/2}, \theta_2(y) = \left[(d^2 - y^2) |\theta_1(y)| \right]^{1/2} \\ \theta_3(y) &= \left[(c^2 - y^2) |\theta_2(y)| \right]^{1/2} \end{aligned} \right\} \quad (4.188)$$

Using numerical techniques, we can easily calculate the integrals shown above.

Then, in (4.158), (4.160), (4.162), (4.164), and (4.170), we get by applying (4.175) - (4.178).

$$\left. \begin{aligned} U_1(x) &= 0, F_1(x) = 0, F_2(x) = 0, F_3(x) = 0, F_4(x) = 0 \\ s_3 &= \frac{1}{3} [(e^2 - c^2)^{3/2} - (e^2 - b^2)^{3/2} - (e^2 - d^2)^{3/2}] \end{aligned} \right\} \quad (4.189)$$

We get and use (4.141) - (4.143) by using (4.184) - (4.188) and the relations (4.140) - (4.143) in (4.145) - (4.152).

$M_1 = M_2 = R_1 = R_2 = N_1 = N_2 = 0$, Then we get,

$$g_{21}(t) = g_{22}(t) = g_{31}(t) = g_{32}(t) = g_{41}(t) = g_{42}(t) = 0 \quad (4.190)$$

$$g_{11}(t) = \frac{2p_0 \Delta_5(t)}{\pi^2 \theta(t)}, \quad b < t < c \quad (4.191)$$

$$g_{12}(t) = \frac{2p_0\Delta_5(t)}{\pi^2\theta(t)}, \quad d < t < e \quad (4.192)$$

$$g_5(t) = \Delta_1(t) + t^2 L_1 + L_2 \quad (4.193)$$

$\Delta_1(t)$ is specified in section 4.187. After using (4.189), we may proceed to evaluate the integrals in (4.157).

$$\sigma_{yy}^{s_3}(x, 0) = \frac{p_0}{\pi} \begin{cases} \frac{\Delta_5(x)}{\theta(x)}, x \in I_1 \\ -\frac{\Delta_5(x)}{\theta(x)}, x \in I_3 \\ \frac{\Delta_5(x)}{\theta(x)}, x \in I_5 \end{cases} \quad (4.194)$$

$$\left. \begin{aligned} \sigma_{yy}^{a_3}(x, 0) &= \sigma_{yy}^{a_4}(x, 0) = 0 \\ \sigma_{xy}^{s_3}(x, 0) &= \sigma_{xy}^{s_4}(x, 0) = \sigma_{xy}^{a_3}(x, 0) = \sigma_{xy}^{a_4}(x, 0) = 0 \end{aligned} \right\} \quad (4.195)$$

Based on the stress-intensity factor defined in (4.172)

$$[K_b, K_c, K_d, K_e] = \frac{p_0}{\pi} \left[\frac{\Delta_5(b)}{n_1(b)}, -\frac{\Delta_5(c)}{n_1(c)}, -\frac{\Delta_5(d)}{n_2(d)}, \frac{\Delta_5(e)}{n_2(e)} \right] \quad (4.196)$$

$$\left. \begin{aligned} n_1(x) &= \sqrt{2x(c^2 - b^2)(d^2 - x^2)(e^2 - x^2)} \\ n_2(x) &= \sqrt{2x(e^2 - d^2)(x^2 - b^2)(x^2 - c^2)} \end{aligned} \right\} \quad (4.197)$$

The following integrals (4.171) and (4.191) - (4.192) will be used numerically to assess the c.o.d.

4.3 TWO WEDGES OPENING FOUR GRIFFITH CRACKS IN ISOTROPIC STRIP STRESS-INTENSITY FACTORS

4.3.1 Introduction

Structures such as the wing or fuselage of an aircraft or a machine often make use of rectangular strips. Adding strip to a machine or building makes it stronger. In reality, the wedges are just solid, hard particles floating about in the medium or the strip. These inflexible bits are byproducts of the production process.

These strips may shatter during prolonged structural usage due to hydrostatic pressure, air pressure, or the high degree of stress buildup at these hard particles or wedges. Mode-I opening describes the process by which fractures open under these conditions.

Zhong et al. (2014) investigated Griffith crack anti-plane shear behavior in functionally graded piezoelectric materials using global theory. We may simplify the issue by reducing it to two integral equations. After that, the dual integral problem was solved using the Schmidt technique. Using critical shear fracture theory as a foundation, Ruiz et al. presented a novel theoretical approach. Shear reinforced slabs' ductility and strength were studied. Neither mode-II nor mode-III nor shear crack theories presents very compelling challenges.

In the present chapter we are going to discuss about Mode-I crack opening by two similar wedges equally spaced from the centre line i.e., y-axis. The cracks occupy the region $y = 0, b < |x| < c, d < |x| < e \leq a$ where $2a$ is the width of strip, see figure 4.3.

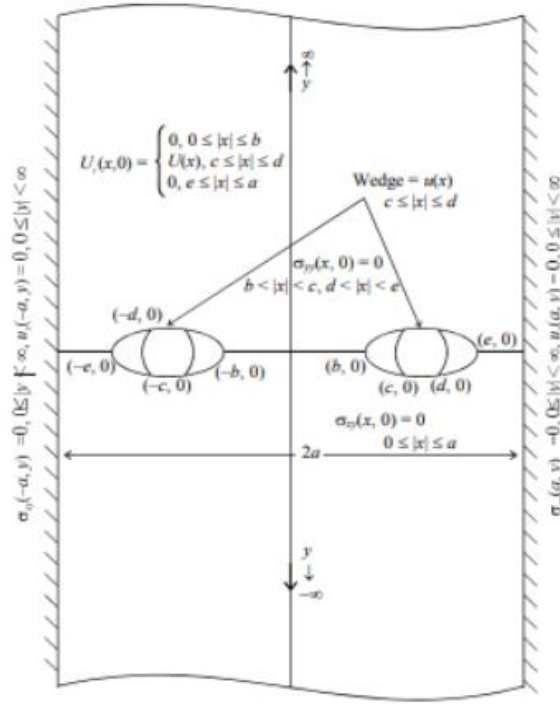


Figure 4.3: Two similar wedges open the cracks in the strip of width $2a$

This reduces the original physical issue to the m-b value problem, which is as

$$\sigma_{xy}(\pm a, y) = 0, \quad 0 \leq |y| < \infty \quad (4.198)$$

$$u_x(\pm a, y) = 0, \quad 0 \leq |y| < \infty \quad (4.199)$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 \leq |x| \leq a \quad (4.200)$$

$$\sigma_{yy}(x, 0) = 0, \quad b < |x| < c, \quad d < |x| < e < a \quad (4.201)$$

$$u_y(x, 0) = \begin{cases} 0, & 0 \leq |x| \leq b \\ u(x), & c \leq |x| \leq d \\ 0, & e \leq |x| \leq a \end{cases} \quad (4.102)$$

When $(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$ and (u_x, u_y) at the medium's general point (x, y) , are components of displacement and stress.

Thus we see that crack faces are stress-free. We checked throughout, see (Burniston E.E., 1969) that

$$u_v(x, 0) > 0, \quad b < |x| < c, \quad d < |x| < e < a \quad (4.203)$$

It may be inferred from this that fractures open completely and, with the exception of their tips, do not contact. The identical procedure as above described can be followed with the finite fourier sine transform with respect to x and the fourier cosine transform with respect to y to perhaps resolve the declining m-b value problem.

$$F_c(\xi) = \int_0^\infty f(y) \cos(\xi y) dy$$

$$F_s(\alpha_n) = \int_0^a f(x) \sin(\alpha_n x) dx$$

With usual inversion formula. $\alpha_n = \frac{n\pi}{a} = nq$. We make use of symmetry of the geometry and placing of equally spaced wedges and positioning of cracks, then the m-b conditions (4.198) – (4.202) are reduced to

$$\sigma_{xy}(a, y) = 0, \quad 0 \leq y < \infty, \quad (4.204)$$

$$u_x(a, y) = 0, \quad 0 \leq y < \infty, \quad (4.205)$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 \leq x \leq a, \quad (4.206)$$

$$\sigma_{yy}(x, 0) = 0, \quad x \in I_2 \cup I_4 \quad (4.207)$$

$$u_y(x,0) = \begin{cases} 0, x \in I_1 \cup I_5 \\ u(x), x \in I_3 \end{cases} \quad (4.208)$$

Where

$$\left. \begin{aligned} I_1 &= [0, b], I_2 = (b, c), I_3 = [c, d] \\ I_4 &= (d, e), I_5 = [e, a] \end{aligned} \right\} \quad (4.209)$$

The solution space is reduced to $[0, a] \cup [0, \infty]$ from $[-a, a] \cup (-\infty, \infty)$. See figure 4.4.

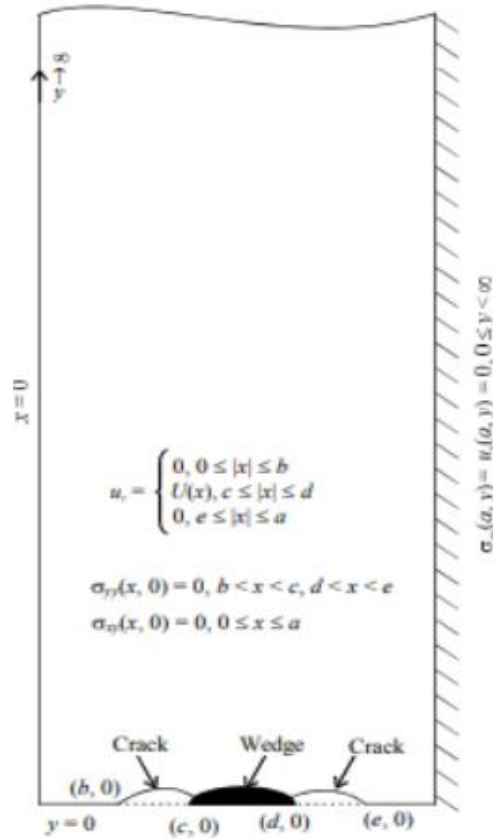


Figure 4.4: Reduced geometry from strip to first quadrant of strip

The issue's blueprint looks like this:

We shall formulate the issue once we present it. We shall provide the answer to the quintuple series problem. After that, we shall talk about the actual amounts. There will

be a function provided that describes the physical quantities. Two unique wedge situations are addressed in the last section.

4.3.2 Formulation

A physical solution based on the equilibrium equations and the systems of equations (4.204)–(4.208)

$$\frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \sigma_{xy} = 0 \quad (4.210)$$

$$\frac{\partial}{\partial x} \sigma_{xy} + \frac{\partial}{\partial y} \sigma_{yy} = 0 \quad (4.211)$$

We follow the method of Airy's stress function (Kushwaha P.S., 1975) and assume the displacement components as

$$u_x(x, y) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin(\alpha_n x)}{\alpha_n} \left[(1 - \eta) \frac{\partial^2 G}{\partial y^2} + \alpha_n^2 \eta G \right] \quad (4.212)$$

And

$$u_y(x, y) = \frac{2}{a} \left[\frac{u_0}{2} + \sum_{n=1}^{\infty} \frac{\cos(\alpha_n x)}{\alpha_n^2} \left[(1 + \eta) \frac{\partial^3 G}{\partial y^3} + (2 + \eta) \alpha_n^2 \frac{\partial G}{\partial y} \right] \right] \quad (4.213)$$

Where

$$u_0 = G(0, y) = A_0 + yB_0 \quad (4.214)$$

$$G(\alpha_n, y) = (A_n + yB_n) e^{-\alpha_n y}, n = 1, 2, \dots \quad (4.215)$$

A and B are two constants that we are attempting to determine. The boundary conditions (4.204-4.205), the equations of equilibrium (4.210-4.211), all the assumptions (4.212-4.213), and the conditions are all met. Below is a list of the stress components.

$$\sigma_{yy}(x, y) = \sum_{n=1}^{\infty} \alpha_n^2 \cos(\alpha_n x) [A_n + yB_n] e^{-\alpha_n y} \quad (4.216)$$

$$\sigma_{xy}(x, y) = - \sum_{n=1}^{\infty} \alpha_n \sin \alpha_n x [-\alpha_n (A_n + yB_n) + B_n] e^{-\alpha_n y} \quad (4.217)$$

The boundary condition (4.206) gives

$$\alpha_n A_n = B_n \quad (4.218)$$

Now the substitution of (4.216) and (4.213) – (4.215) gives the quintuple series equation as

$$\frac{u_0}{2} + \sum_{n=1}^{\infty} B_n \cos(\alpha_n x) = \begin{cases} 0, & x \in I_1 \\ u(x), & x \in I_3 \\ 0, & x \in I_5 \end{cases} \quad (4.219)$$

$$\left. \begin{aligned} \alpha_0 &= \frac{aE}{2(1+\eta)} \\ u_1(x) &= \frac{2}{a} \alpha_0 u(x) \end{aligned} \right\} \quad (4.220)$$

$$\sum_{n=1}^{\infty} \alpha_n B_n \cos \alpha_n x = 0, x \in I_2 \cup I_4 \quad (4.221)$$

The equations (4.219) – (4.221) are called quintuple series equation. Therefore, solving the quintuple series equation above reduces the m-b value issue to a single variable. In next section we will solve the above series equation.

4.3.3 Solution of Quintuple Series Equation

Using the approach of (Kushwaha P.S., 1975), we may get the solutions of (4.219) - (4.220). Assuming the solution of the quintuple series equation is as

$$B_n = \frac{2}{\alpha_n} \left[\left\langle \int_b^c g(t) + \int_d^e h(t) \right\rangle \sin \alpha_n t dt - \frac{1}{a} \int_c^d u_1'(t) \sin \alpha_n t dt \right], \quad (4.222)$$

$$u_1 = 2 \left[\left(\int_b^c t g(t) + \int_d^e t h(t) - \frac{1}{a} \int_c^d u_1'(t) \right) dt \right], \quad (4.223)$$

Thus using the relation

$$\frac{t}{2} + \sum_{n=1}^{\infty} \frac{\sin nt \cos nx}{n} = \begin{cases} \frac{\pi}{2}, & t > x \\ \frac{\pi}{4}, & t = x \\ 0, & t < x \end{cases} \quad (4.224)$$

The substitution of (4.222) (4.223) into (4.219) and then using (4.224), the equation is satisfied identically if

$$\int_b^c g(t) dt = -\frac{1}{a} u_1(c) \quad (4.225)$$

$$\int_d^e h(t) dt = \frac{1}{a} u_1(d) \quad (4.226)$$

Now we substitute (4.222) into (4.220) and using relation

$$\sum_{n=1}^{\infty} \frac{\sin(\alpha_n t) \sin(\alpha_n x)}{\alpha_n} = \frac{1}{2q} \log \left| \frac{\sin q \left(\frac{x-t}{2} \right)}{\sin q \left(\frac{x+t}{2} \right)} \right| \quad (4.227)$$

And then using the method of (Kushwaha P.S., 1975) We get

$$g(t) = \frac{2}{a^2} \frac{\Delta_1(t)}{\delta(t)}, \quad b < t < c \quad (4.228)$$

$$h(t) = \frac{2}{a^2} \frac{\Delta_1(t)}{\delta(t)}, \quad d < t < e \quad (4.229)$$

$$\Delta_1(t) = \left[\left(\int_b^c - \int_d^e \right) \frac{\sin(qx)\delta(x)P_0(x)dx}{G(x,t)} + M_1 \cos(qt) + M_2 \right] \quad (4.230)$$

With

$$G(x,t) = |\cos(qx) - \cos(qt)| \quad (4.231)$$

$$\delta(x) = [G(x,b)G(x,c)G(x,d)G(x,e)]^{1/2}, \quad (4.232)$$

$$P_0(x) = \int_c^d \{ \sin(at)u_1'(t) / G(x,t) \} dt \quad (4.233)$$

Steps (4.225)–(4.226) and (4.228)–(4.229) must be fulfilled in order to get two random constants, M1 and M2. Simplification of the physical problem can be done by deciding the functions. $g(t)$ and $h(t)$ of various forms of. $u(x)$, the wedge profile function. After that, the stress function and the estimated COD (crack opening displacement) are used to develop closed form equations.

4.3.4 Physical Quantities

The physical quantities which are important in fracture theory are displacement component $u_y(x,0)$ and $\sigma_{yy}(x,0)$ normal stress in the nbd of crack tips.

Crack Shape

The displacement graph represents the fracture form. $u_y(x,0), x \in I_2 \cup I_4$. Using the series values on the left side of (4.219) and the relations (4.222) - (4.223), this expression is evaluated.

$$u_y(x,0) = \alpha_0^{-1} \begin{cases} \int_x^c g(t)dt + \frac{1}{a}u_1(c) \\ \int_x^e h(t)dt \end{cases} \quad (4.234)$$

If you subtract (4.229) from (4.228), you get $g(t)$ and $h(t)$.

Stress components

The shear stress over $y = 0$ is zero. The normal stress which is in opening mode of σ crack is $\sigma_{yy}(x,0)$. Therefore, we shall evaluate this stress component from (4.216) for $y = 0$.

$$\sigma_{yy}(x,0) = \sum \alpha_n^2 \cos(\alpha_n x) A_n = \sum \alpha_n B_n \cos(\alpha_n x) \quad (4.235)$$

After using (4.218). Now we use (4.222) in (4.235) we get,

$$\begin{aligned} \sigma_{yy}(x,0) &= 2 \left\langle \left(\int_b^c g(t) + \int_d^e h(t) - \frac{1}{a} \int_c^d u_1'(t) \right) \sum \sin \alpha_n t \cos \alpha_n x dt \right\rangle \\ \sigma_{yy}(x,0) &= \left\langle \left(\int_b^c g(t) + \int_d^e h(t) \right) \frac{\sin(qt)}{G(x,t)} dt - \frac{1}{a} \int_c^d \frac{u_1(t) \sin(qt)}{G(x,t)} dt \right\rangle \end{aligned} \quad (2.236)$$

Now using (4.228) and (4.229) in (4.236) and evaluating the integrals we get

$$\sigma_{yy}(x,0) = \frac{2}{a} \begin{cases} \frac{\Delta_1(x)}{\delta(x)} - P_0(x), & x \in I_1 \\ -\frac{\Delta_1(x)}{\delta(x)} - P_0(x), & x \in I_3 \\ \frac{\Delta_1(x)}{\delta(x)} - P_0(x), & x \in I_5 \end{cases} \quad (4.237)$$

With

$$\Delta_1(x) = \left(\int_b^c - \int_d^e \right) \frac{\sin(qy) \delta(y) P_0(y) dy}{G(y, x)} + M_1 \cos(qx) + M_2 \quad (4.238)$$

And $\delta(x)$ is defined in (4.231).

Stress Intensity Factors

S.I.F. at crack-tips are characterized by:

$$\left. \begin{aligned} K_b &= \lim_{x \rightarrow b^-} \sqrt{b-x} \sigma_{yy}(x, 0) \\ K_c &= \lim_{x \rightarrow c^+} \sqrt{x-c} \sigma_{yy}(x, 0) \\ K_d &= \lim_{x \rightarrow d^-} \sqrt{b-x} \sigma_{yy}(x, 0) \\ K_e &= \lim_{x \rightarrow e^-} \sqrt{b-e} \sigma_{yy}(x, 0) \end{aligned} \right\} \quad (4.239)$$

We use (4.237) in (4.239) and evaluate the limits we get

$$\left. \begin{aligned} K_b &= \frac{2\Delta_1(b)}{\delta_1(b)}, \\ K_c &= -\frac{2\Delta_1(c)}{\delta_2(e)}, \end{aligned} \right\} \quad (4.240)$$

$$\left. \begin{aligned} K_d &= +\frac{2\Delta_1(d)}{\delta_3(d)}, \\ K_e &= -\frac{2\Delta_1(e)}{\delta_4(e)}, \end{aligned} \right\} \quad (4.241)$$

$$\left. \begin{aligned} \delta_1(b) &= [\pi a \sin(qb) G(b,c) G(b,d) G(b,e)]^{\frac{1}{2}} \\ \delta_2(c) &= [\pi a \sin(qc) G(b,c) G(c,d) G(c,e)]^{\frac{1}{2}} \\ \delta_3(d) &= [\pi a \sin(qd) G(b,d) G(c,d) G(d,e)]^{\frac{1}{2}} \\ \delta_4(e) &= [\pi a \sin(qe) G(b,e) G(c,e) G(d,e)]^{\frac{1}{2}} \end{aligned} \right\} \quad (4.242)$$

Thus the problem is reduced to the evaluation of $g(t)$ or $\Delta_1(t)$. In next section we shall evaluate this.

4.3.5 Evaluation of G(T) For General Shape of Wedge

The evaluation of crack shape depends upon the evaluation of $g(t)$ and $h(t)$ which depends upon $\Delta_1(t)$. $g(t)$ and $h(t)$ both involves same constants M_1 and M_2 . Thus we evaluate $\Delta_1(t)$ as

$$\Delta_1(t) = \left[\frac{a}{2} \{ L_0 + G(b,t)L_1 + G(b,t)G(t,c)L_2 + G(b,t)G(t,c)G(t,d)L_3 \} + M_1 \cos(qt) + M_2 \right] \quad b < t < c \quad (4.243)$$

with

$$\left. \begin{aligned} L_0 &= \left(\int_b^c - \int_d^e \right) \frac{\sin(qx) \delta(x) P_0(x)}{G(b,x)} dx \\ L_1 &= \left(\int_b^c - \int_d^e \right) \frac{\sin(qx) \delta(x) P_0(x)}{G(b,x) G(x,c)} dx \\ L_2 &= \left(\int_b^c - \int_d^e \right) \frac{\sin(qx) \delta(x) P_0(x)}{G(b,x) G(x,c) G(x,d)} dx \\ L_3 &= \left(\int_b^c - \int_d^e \right) \frac{\sin(qx) P_0(x)}{\delta(x)} dx \end{aligned} \right\} \quad (4.244)$$

We use (4.243) - (4.244), (4.228) - (4.229), and (4.225) - (4.226) to get the values of constants M_1 and M_2 , and we compute the integrals for exceptional loads, but otherwise we employ numerical methods.

$$L_0N_1 + L_1N_2 + L_2N_3 + L_3N_4 + M_1N_5 + M_2N_1 = \frac{1}{2}u_1(c) = u_2 \quad (4.245)$$

$$L_4N_6 + L_5N_7 + L_6N_8 + L_7N_9 + M_1N_{10} + M_2N_6 = \frac{u_1(d)}{a} \quad (4.246)$$

$$N_1 = \int_b^c \frac{dt}{\delta(t)}, N_2 = \int_b^c \frac{G(b,t)dt}{\delta(t)}, N_3 = \int_b^c \frac{G(b,t)G(t,c)dt}{\delta(t)} \quad (4.247)$$

$$N_4 = \int_b^c \frac{G(b,t)G(t,c)G(t,d)dt}{\delta(t)}, N_5 = \int_d^c \frac{\cos(qt)dt}{\delta(t)} \quad (4.248)$$

$$N_6 = \int_e^e \frac{dt}{\delta(t)}, N_7 = \int_d^e \frac{G(b,t)dt}{\delta(t)}, N_8 = \int_d^e \frac{G(b,t)G(c,t)dt}{\delta(t)} \quad (4.249)$$

$$N_9 = \int_d^e \frac{G(b,t)G(c,t)G(d,t)dt}{\delta(t)}, N_{10} = \int_d^e \frac{\cos(qt)dt}{\delta(t)} \quad (4.250)$$

We first express the equations as (4.245) - (4.246) before solving them for M1 and M2.

$$\left. \begin{aligned} M_1N_5 + M_2N_1 &= u_2 - R_1 = R_2 \\ M_1N_{10} + M_2N_6 &= -R_3 + \frac{u_1(d)}{a} = R_4 \end{aligned} \right\}$$

$$\text{Then solving,} \quad (4.251)$$

$$M_1 = \left(\frac{R_2N_6 + R_4N_1}{R_5} \right) \quad (4.252)$$

$$M_2 = - \left(\frac{R_2 N_{10} + R_4 N_5}{R_5} \right)$$

where

(4.253)

$$\left. \begin{aligned} R_1 &= L_0 N_1 + L_1 N_2 + L_2 N_3 + L_3 N_4, R_2 = u_2 - R_1 \\ R_3 &= L_4 N_6 + L_5 N_7 + L_6 N_8 + L_7 N_9 \\ R_5 &= N_5 N_6 - N_{10} N_1, R_4 = \frac{u_1(d)}{a} - R_3 \end{aligned} \right\} \quad (4.254)$$

Therefore, the physical quantities may be simply evaluated with knowledge of $\Delta 1$ (t).

4.3.6 A Particular Sort of Wedge

In the first case, we look at figures 4.5–4.6, which show a rectangular wedge with dimensions $(d - c)$ and $2h$ for width.

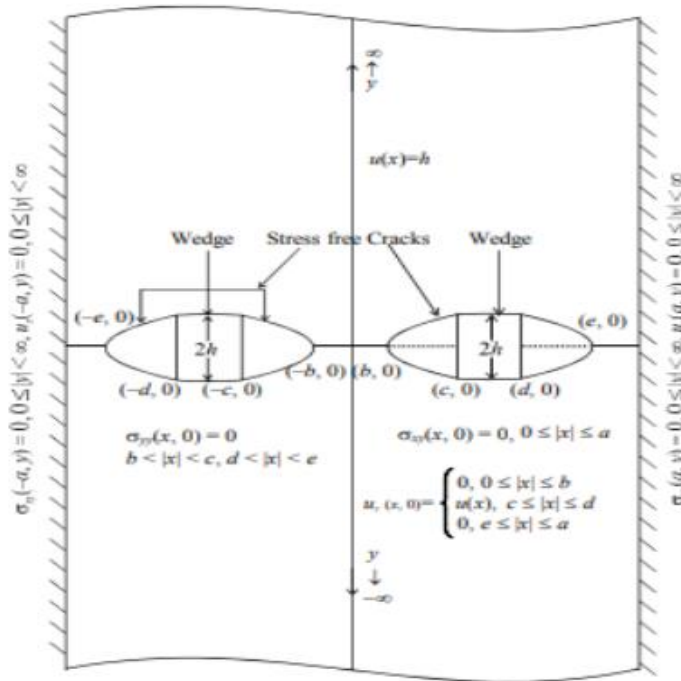


Figure 4.5: Cracks are opened by rectangular wedges of length $(d - c)$ and of width $2h$

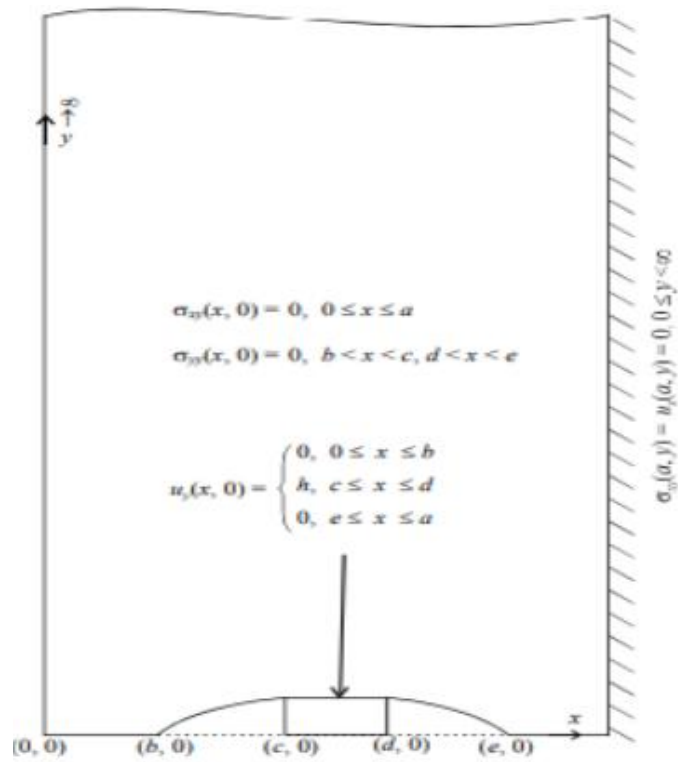


Figure 4.6: Geometry in first quadrant of strip with rectangular wedge

$$u(x) = h = \text{constant}$$

and

(4.255)

$$u_1(x) = h\alpha_0, \quad \alpha_0 = \frac{aE}{2(1+\eta)}$$

(4.256)

Then

$$u_1'(x) = 0$$

Thus

(4.257)

$$P_0(x) = 0$$

(4.258)

$$\text{Then } \left. \begin{array}{l} L_i = 0, i = 0, \dots, 9 \\ R_2 = \alpha_0 / 0, R_1 = h\alpha_0 / a \end{array} \right\} \quad (4.259)$$

The evaluation of constants M_1 and M_2 will lead to

$$M_1 = \frac{(-N_6 + N_1)h\alpha_0}{aR_5} \quad (4.260)$$

$$M_2 = + \frac{(N_{10} + N_5)h\alpha_0}{aR_5}$$

$$\text{Thus} \quad (4.261)$$

Thus

$$\left. \begin{array}{l} \Delta_1(t) = M_1 \cos(qt) + M_2, \quad b < t < c \\ \Delta_1(t) = -[M_1 \cos(qt) + M_2], \quad d < t < e \end{array} \right\} \quad (4.262)$$

Since this is the case, we can readily use (4.262), (4.228)–(4.229), (4.234), and (4.240)–(4.241) to compute the numerical crack shape and S.I.F.

Situation 2: Here, we take a trapezium wedge into account (see to figures 4.7 and 4.8). From points (c,0) and (d,0), we take into account the wedge with length (d-c) and heights $2h_1$ and $2h_2$, respectively (Fig. 4.7 and 4.8).

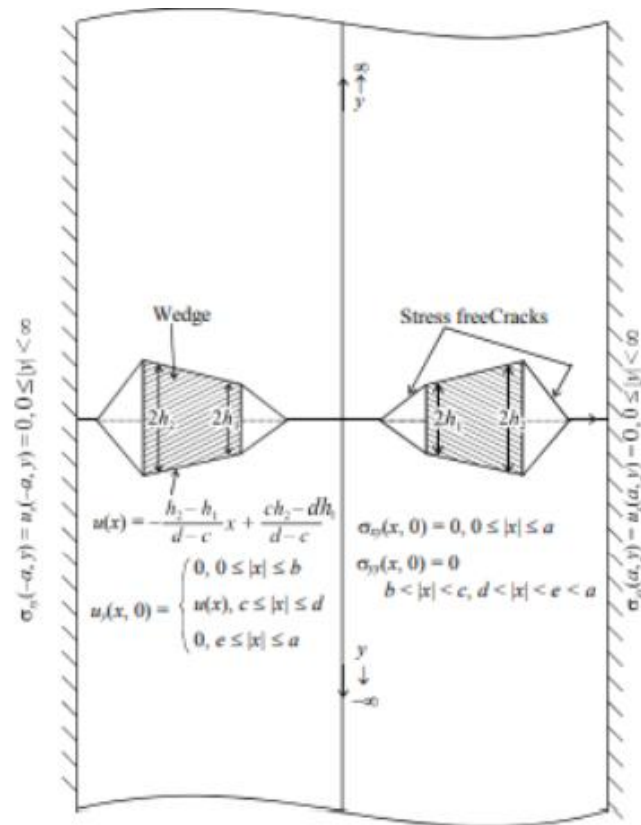


Figure 4.7: Trapezium type of wedges open the cracks

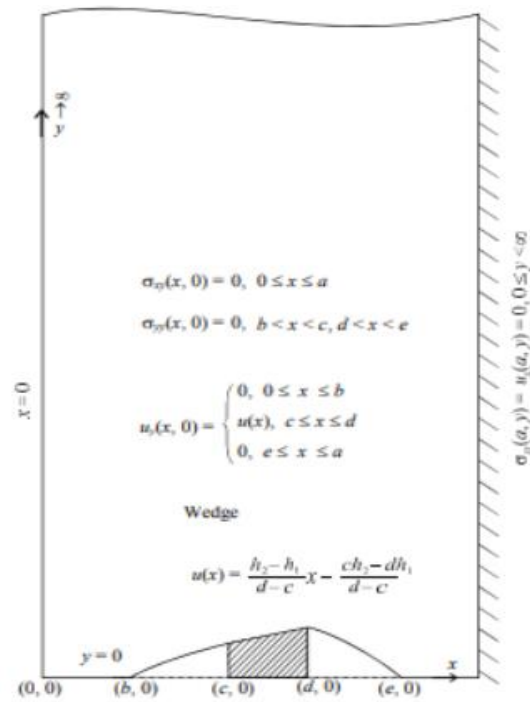


Figure 4.8: Geometry of problem in first quadrant of strip

And $h_2 > h_1$.

$$\left. \begin{aligned} u(x) &= \frac{h_2 - h_1}{d - c} x - \frac{ch_2 - dh_1}{d - c} \\ \text{and } u(x) &= -\frac{h_2 - h_1}{d - c} x + \frac{ch_2 - dh_1}{d - c} \end{aligned} \right\}$$

Thus (4.263)

$$u'(x) = \frac{h_2 - h_1}{d - c} \quad (4.264)$$

$$P_0(x) = \alpha_1 \log \left| \frac{G(c, x)}{G(x, d)} \right|, \quad x \in I_2 \cup I_4, \quad (4.265)$$

$$\alpha_1 = \frac{h_2 - h_1}{d - c} \cdot \frac{aE}{2(1 + \eta)} \quad (4.266)$$

Substitute (4.245) and (4.246) for (4.265) and (4.266), respectively, and compute the integrals. After then, get $\Delta 1(t)$ from (4.243) by following (4.247) - (4.254).

4.4 TWO SIMILAR WEDGES OPEN FOUR GRIFFITH CRACKS IN ISOTROPIC RECTANGLE

4.4.1 Introduction

What you see as wedges in the medium are really hard particles that were leftover from the production process. The square forms a dish. Making machinery or aeroplanes is a common usage for the plates.

The initial physical difficulty is simplified by this m-b value problem. On the x-axis, the wedges are visible. Check see figure 4.9.

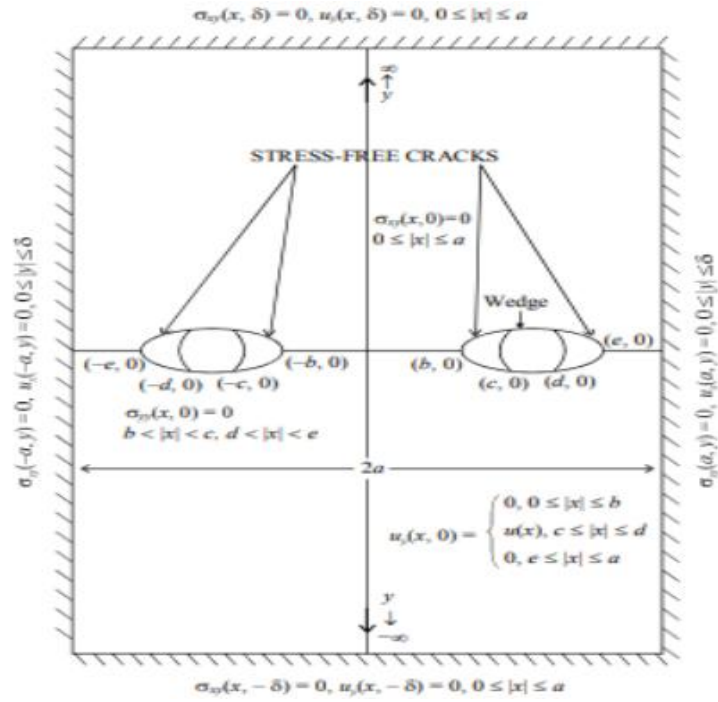


Figure 4.9: Geometry of physical problem

$$\sigma_{xy}(\pm a, y) = 0, \quad 0 \leq |y| \leq \delta \quad (4.267)$$

$$\sigma_{xy}(x, \pm \delta) = 0, \quad 0 \leq |x| \leq a \quad (4.268)$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 \leq |x| \leq a \quad (4.269)$$

And

$$u_x(\pm a, y) = 0, \quad 0 \leq |y| \leq \delta \quad (4.270)$$

$$u_y(x, \pm \delta) = 0, \quad 0 \leq |x| \leq a \quad (4.271)$$

Where $2a$ is length and 2δ is width of the rectangle, see figure 4.9. The mixed boundary conditions are as

$$u_y(x,0) = \begin{cases} 0, & 0 \leq |x| \leq b \\ u(x), & c \leq |x| \leq d \\ 0, & e \leq |x| \leq a \end{cases} \quad (4.272)$$

$$\sigma_{yy}(x,0) = 0, \quad b < |x| < c, \quad d < |x| < e \quad (4.273)$$

Where $u(x)$ is shape function of wedge. Cracks are of lengths $(c - b)$, $(e - d)$ while cracks occupy equal distance from y -axis at x -axis. We are assuming the plane-strain conditions to prevail. There is geometrical as well as material symmetry. Therefore, the solution space $[-a, a] \times [-\delta, \delta]$ is reduced to $[0, a] \times [0, \delta]$ i.e. first quadrant of

rectangle. Then the boundary condition (4.267) - (4.273) are reduced to

$$\sigma_{xy}(a, y) = 0, \quad 0 \leq y \leq \delta, \quad (4.274)$$

$$\sigma_{xy}(x, \delta) = 0, \quad 0 \leq x \leq a, \quad (4.275)$$

$$\sigma_{xx}(x, 0) = 0, \quad 0 \leq x \leq a, \quad (4.276)$$

And

$$u_x(a, y) = 0, \quad 0 \leq y \leq \delta, \quad (4.277)$$

$$u_y(x, \delta) = 0, \quad 0 \leq x \leq a, \quad (4.278)$$

$$u_y(x,0) = \begin{cases} 0, & 0 \leq x \leq b \\ u(x), & c < x < d \\ 0, & e \leq x \leq a \end{cases} \quad (4.279)$$

$$\sigma_{yy}(x,0) = 0, \quad b < x < c, \quad d < x < e \quad (4.280)$$

We observed throughout, see Butniston [45]

$$u_y(x,0) > 0, \quad b < |x| < c, \quad d < |x| < e \quad (4.281)$$

4.4.2 Formulation

The aforementioned mathematical answer may be obtained by solving the equilibrium equations in (4.274) through (4.279).

$$\left. \begin{aligned} \frac{\partial}{\partial x} \sigma_{xx}(x,y) + \frac{\partial}{\partial y} \sigma_{xy}(x,y) &= 0 \\ \frac{\partial}{\partial x} \sigma_{xy}(x,y) + \frac{\partial}{\partial y} \sigma_{yy}(x,y) &= 0 \end{aligned} \right\} \quad (4.282)$$

with the following S-S relations as

$$\sigma_{ij}(x,y) = 2e_{ij}(x,y) + \frac{2(1+\eta)}{1-2\eta} e_{kk}(x,y) \delta_{kk}; \quad i, j = x, y \quad (4.283)$$

The stress units are μ (units of rigidity) with η being the Poisson ratio, and E being the Young's modulus of the medium.

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\left. \begin{aligned} e_{ij} &= u_{i,j} + u_{j,i} \\ e_{kk} &= u_{x,x} + u_{y,y} \end{aligned} \right\} \quad (4.284)$$

Differentiation with regard to the letters written over there is denoted by the symbol (,) over quantities. Components e_{ij} represent strain, u_i represent displacement, and σ_{ij} represent stress. By using Airy's function approach, we may get the solution of (4.282). Presumably, the answer to the equation (4.282) is

$$u_x(x, y) = \frac{2(1+\eta)}{E} \sum_{n=1}^{\infty} \frac{\sin \alpha_n x}{\alpha_n} \left[(1-\eta)G_{,yy} + \eta \alpha_n^2 G_{,y} \right] \quad (4.285)$$

$$u_y(x, y) = \frac{2(1+\eta)}{E} \left[\frac{1}{2} u_{yc}(0, y) + \sum_{n=1}^{\infty} \frac{\cos \alpha_n x}{\alpha_n^2} u_{yc}(\alpha_n, y) \right] \quad (4.286)$$

with

$$u_{yc}(\alpha_n, y) = (A_n + yB_n) \cosh(\alpha_n y) + (C_n + yD_n) \sinh(\alpha_n y)$$

$$\alpha_n = \frac{n\pi}{a} = nq. \quad (4.287)$$

And $A_n, B_n; C_n, D_n$ are four sets of constants to be determined. Thus the displacement and stress components are given below:

$$u_x(x, y) = \frac{2(1+\eta)}{E} \sum_{n=1}^{\infty} \sin(\alpha_n x) \left[\alpha_n (A_n + yB_n) \cosh(\alpha_n y) + \alpha_n (C_n + yD_n) \sinh \alpha_n y \right. \\ \left. + 2(1-\eta) \{ B_n \sinh \alpha_n y + D_n \cosh \alpha_n y \} \right] \quad (4.288)$$

$$u_{yc}(\alpha_n, y) = \sum_{n=1}^{\infty} \cos(\alpha_n x) \left[\alpha_n (A_n + yB_n) \sinh(\alpha_n y) + \alpha_n (C_n + yD_n) \cosh(\alpha_n y) \right. \\ \left. + (3-2\eta) \{ B_n \cosh(\alpha_n y) + D_n \sinh(\alpha_n y) \} \right] \quad (4.289)$$

$$u_{yc}(0, y) = (3-2\eta)B_0 \quad (4.290)$$

$$\sigma_{yy}(x, y) = - \sum_{n=1}^{\infty} \alpha_n \cos \alpha_n x \left[\alpha_n (A_n + yB_n) \cosh(\alpha_n y) \right.$$

$$+ \alpha_n (C_n + yD_n) \sinh(\alpha_n y) + 2B_n \sinh(\alpha_n y) + 2D_n \cosh(\alpha_n y) \quad (4.291)$$

$$\sigma_{xy}(x, y) = \sum_{n=1}^{\infty} \alpha_n \sin \alpha_n x \left[\alpha_n (A_n + yB_n) \sinh(\alpha_n y) \right. \\ \left. + \alpha_n (C_n + yD_n) \cosh(\alpha_n y) + B_n \cosh(\alpha_n y) + D_n \sinh(\alpha_n y) \right] \quad (4.292)$$

$$+ \alpha_n (C_n + yD_n) \sinh(\alpha_n y) + 2B_n \sinh(\alpha_n y) + 2D_n \cosh(\alpha_n y) \quad (4.293)$$

The constants A_n, B_n, C_n, D_n will be determined in next section.

4.4.3 Reduction to Quintuple Series Equation

The following relations will be obtained from the border and m-b conditions (4.274) - (4.279) and (4.288) - (4.292). Using the boundary conditions from sections 4.275, 4.276, and 4.278, the following relations will be obtained:

$$B_n = -\alpha_n C_n \quad (4.294)$$

$$B_n = -D_n \tanh(\alpha_n \delta) \quad (4.295)$$

$$\alpha_n C_n = D_n \tanh(\alpha_n \delta)$$

and (4.296)

$$\alpha_n A_n = D_n \left[\frac{2\alpha_n \delta - \sinh(2\alpha_n \delta)}{\sinh(2\alpha_n \delta)} \right], \quad (4.297)$$

Thus out of four sets of constants three i.e. B_n , C_n and A_n , can be reduced to in terms of D_n . The m-b conditions (4.279) and (4.280), after using (4.286), (4.289) and (4.291) along with (4.294) - (4.297). We get,

$$\frac{\phi_0}{2} - 2(1-\eta) \sum_{n=1}^{\infty} D_n \cos(\alpha_n x) \tanh(\alpha_n \delta) = \begin{cases} 0, & 0 \leq x \leq b \\ u_1(x), & c \leq x \leq d \\ 0, & e \leq x \leq a \end{cases} \quad (4.298)$$

and

$$\sum_{n=1}^{\infty} \alpha_n D_n \cos(\alpha_n x) \left[\frac{\sinh 2\alpha_n \delta + 2\alpha_n \delta}{\sinh 2\alpha_n \delta} \right] = 0, b < x < c, d < x < e \quad (4.299)$$

$$\phi_0 = \frac{(3-2\eta)B_0}{2}, u_1(x) = \frac{Eq}{2(1+\eta)} \cdot u(x) \quad (4.300)$$

Now we assume that,

$$D_n \tanh \alpha_n \delta = \phi_n \quad (4.301)$$

Thus (4.298) - (4.299) become

$$\frac{\phi_0}{2} + \sum_{n=1}^{\infty} \phi_n \cos(\alpha_n x) = \begin{cases} 0, & 0 \leq x \leq b \\ u_1(x), & c \leq x \leq d \\ 0, & e \leq x \leq a \end{cases} \quad (4.302)$$

And

$$\sum_{n=1}^{\infty} \alpha_n \phi_n \cos(\alpha_n x) = -p(x), \quad b < x < c, \quad d < x < e \quad (4.303)$$

$$p(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n M(\alpha_n \delta) \cos(\alpha_n x), \quad (4.304)$$

$$M(\alpha_n \delta) = e^{-\alpha_n \delta} \operatorname{cosech}(\alpha_n \delta) + \alpha_n \delta \operatorname{cosech}^2(\alpha_n \delta), \quad (4.305)$$

The only remaining physical problem is to solve the quintuple series equations (4.302) - (4.305).

4.4.4 Solution of Quintuple Series Equation

To solve the quintuple series problem, we will use the approach of (Kushwaha P.S., 1975). Given this, we will treat the answer to (4.302) as

$$\phi_n = \frac{1}{2\alpha_n} \left[\left\langle \int_b^c g(g) + \int_d^e h(t) \right\rangle \sin(\alpha_n t) dt - \frac{2}{a} \int_c^d u'_1(t) \sin \alpha_n t dt \right], \quad (4.306)$$

And

$$\phi_0 = \frac{1}{2} \left[\int_b^c t g_0(t) dt + \int_d^e h(t) dt - \frac{2}{a} \int_c^d t u'_1(t) dt \right] \quad (4.307)$$

Now substituting (4.305) – (4.307) into (4.302) and using the relation

$$\sum_{n=1}^{\infty} \frac{\sin(\alpha_n t) \cos(\alpha_n x)}{n} = \begin{cases} \frac{a-t}{2}, & t > x \\ \frac{a-t}{4}, & t = x \\ -\frac{t}{2}, & t < x \end{cases} \quad (4.308)$$

Then it satisfies the same if

$$\int_b^c g_0(t)dt = \frac{1}{a}u_1(c) \quad (4.309)$$

$$\int_d^e h(t)dt = \frac{1}{a}u_1(d) \quad (4.310)$$

We substitute (4.306) into (4.303) and using the relation

$$\sum_{n=1}^{\infty} \frac{\sin(\alpha_n t) \sin(\alpha_n x)}{n} = \frac{1}{2} \log \left| \frac{\sin q \frac{(x-t)}{2}}{\sin q \frac{(x+t)}{2}} \right| \quad (4.311)$$

And then using inversion formula of (Kushwaha P.S., 1975)

$$g(t) = \frac{2}{a^2 \theta(t)} \left[\Delta(t) + \left\langle \int_b^c g(y) - \int_d^e h(y) \right\rangle K(y, t) dy \right], b < t < c \quad (4.312)$$

$$h(t) = \frac{2}{a^2 \theta(t)} \left[\Delta(t) + \left\langle \int_b^c g(y) - \int_d^e h(y) \right\rangle K(y, t) dy \right], d < t < e \quad (4.313)$$

With

$$\Delta(t) = \Delta_0(t) + \cos(qt)M_1 + M_2 \quad (4.314)$$

$$\Delta_0(t) = \left(\int_b^c - \int_d^e \right) \frac{p_1(x) \theta(x) \sin(qx) dx}{G(x, t)} \quad (4.315)$$

$$p_1(x) = \int_c^d \frac{u'_1(y) \sin(qy) dy}{G(y, x)}, \quad (4.316)$$

$$K(x, t) = \sum_{n=1}^{\infty} \sin(\alpha_n t) \cos(\alpha_n x) M(\alpha_n \delta) \quad (4.317)$$

Two random constants, M1 and M2, are to be found via (4.309) - (4.310). $M(\alpha_n \delta)$ is given in (4.305). $\theta(x)$ is given below

$$\theta(x) = \left[G(b, x)G(x, c)G(x, d)G(x, e) \right]^{1/2} \quad (4.318)$$

$$G(b, x) = |\cos(qb) - \cos(qx)| \quad (4.319)$$

This reduces the complex physical problem to a connected second-kind Fredholm-integral equation (IE) given by (4.312) - (4.319) in terms of quintuple series equations. In what follows, we evaluate the material variables.

4.4.5 Physical Quantities

Important parameters in fracture mechanics include the critical operating distance (c.o.d.) and the stress component in the viscosity of crack tips.

Crack Shape

An evaluation of the crack's form will be done using the c.o.d. value, and an evaluation of the crack's opening will be done using the value of $u_y(x, 0)$ from series (4.302) for $b < x < c$, $d < x < e$

$$u_y(x, 0) = \frac{2(1 + \eta)}{E} \begin{cases} \int_x^c g(t)dt + \frac{1}{a}u_1(c), b < x < c \\ \int_x^e h(t)dt, d < x < e \end{cases} \quad (4.320)$$

We can get $g_1(t)$ and $h(t)$ by solving the Fredholm integral equations (IE) (4.312) through (4.313).

Stress elements

Throughout, the shear tension at $y=0$ is zero. $0 \leq |x| \leq a$. So, the important stress components is $\sigma_{yy}(x,0)$. This component is obtained through the value of series in (4.303) while right hand side is taken to left and side.

$$\sigma_{yy}(x,0) = \sum_{n=1}^{\infty} \alpha_n \phi_n \cos \alpha_n x + \sum_{n=1}^{\infty} \alpha_n \phi_n M(\alpha_n \delta) \cos \alpha_n x \quad (4.321)$$

We use (4.306) in (4.321) and then use (4.312) - (4.313) and evaluate the integrals

$$\begin{aligned} \sigma_{yy}(x,0) = & \frac{2}{q} \left[\left\langle \int_b^c g(t) + \int_d^e h(t) - \frac{1}{a} \int_e^d u'_1(t) \right\rangle \frac{\sin qt}{G(x,t)} dt \right. \\ & \left. + \left(\left\langle \int_b^c g(t) + \int_d^e h(t) - \frac{1}{a} \int_c^d u'_1(t) \right\rangle K(x,t) \right) dt \right] \end{aligned} \quad (4.322)$$

Thus evaluating the integral, after using (4.312) - (4.313), we get

$$\sigma_{yy}(x,0) = \frac{2}{a^2} \begin{cases} \frac{\Delta_1(x)}{\theta(x)} + F_1(x), 0 \leq x < b \\ -\frac{\Delta_1(x)}{\theta(x)} + F_1(x), c < x < d \\ \frac{\Delta_1(x)}{\theta(x)} + F_1(x), e < x \leq a \end{cases} \quad (4.323)$$

With,

$$\Delta_1(x) = \Delta(x) + \left\langle \int_b^c g(y) - \int_d^e h(y) \right\rangle K(y,x) dy \quad (4.324)$$

$$F_1(x) = \left(\left\langle \int_b^c g(y) - \int_d^e h(t) - \frac{1}{a} \int_c^d u'_1(t) \right\rangle K(x,t) dt \right) \quad (4.325)$$

Stress-Intensity Factors

The S.I.F. are defined as

$$K_b = \lim_{x \rightarrow b^+} \sqrt{b-x} \sigma_{yy}(x, 0), \quad (4.326)$$

$$K_c = \lim_{x \rightarrow c^-} \sqrt{x-c} \sigma_{yy}(x, 0), \quad (4.327)$$

$$K_d = \lim_{x \rightarrow d^+} \sqrt{d-x} \sigma_{yy}(x, 0), \quad (4.328)$$

$$K_e = \lim_{x \rightarrow e^-} \sqrt{x-e} \sigma_{yy}(x, 0), \quad (4.329)$$

The values of $\sigma_{yy}(x, 0)$ for different values of x are used from (4.323) into (4.326) - (4.329) and evaluated the limits.

$$K_b = \frac{1}{a} \frac{\Delta_1(b)}{\sqrt{q \sin qb G(b, c) G(b, d) G(b, e)}} \quad (4.330)$$

$$K_c = - \frac{\Delta_1(c)}{a \sqrt{q \sin(qc) G(b, c) G(c, d) G(c, e)}} \quad (4.331)$$

$$K_d = - \frac{\Delta_1(d)}{a \sqrt{q \sin(qd) G(b, d) G(c, d) G(d, e)}} \quad (4.332)$$

$$K_e = \frac{\Delta_1(e)}{a \sqrt{q \sin(qe) G(b, e) G(c, e) G(d, e)}} \quad (4.333)$$

Where $\Delta_1(x)$ is given in (4.4.5.5).

4.4.6 The Fredholm Integral equation (IE) and Its Solution

First we will expand the Kernel $K(y, t)$ given by (4.317) in terms of $e^{-\alpha_n \delta}$ few terms because the series is very fast decreasing series. and retain

Thus

$$M(\alpha_n \delta) = 6 \sum_{r=1}^{\infty} e^{-2r\alpha_n \delta} \square 6 \sum_{r=1}^2 e^{-2r\alpha_n \delta} \quad (4.334)$$

Then using the relation of series,

$$\sum_{n=1}^{\infty} e^{-n\delta} \sin nx = \frac{\sin x}{2(\cosh \delta - \cos x)} \quad (4.335)$$

We get,

$$K(x, t) = \frac{3}{2} \sum_{r=1}^2 R^{\pm}(\delta, x \pm t) \quad (4.336)$$

With

$$R^{\pm}(\delta, x \pm t) = \frac{\sin q(x+t)}{\cosh 2rq\delta - \cos q(x+t)} - \frac{\sin q(x-t)}{\cosh 2rq\delta - \cos q(x-t)} \quad (4.337)$$

Then, using the binomial expansion provided, extend the right-hand side of (4.337) after maintaining just the values up to

$$R^{\pm}(\delta, x \pm t) = \frac{1}{\cosh 2rq\delta} \left[2 \sin 2qt \cos 2qx + \frac{1 \sin 4qt \cos 4qx}{\cosh 2rq\delta} \right] = R(\delta; x, t) \quad (4.338)$$

In equation (4.336), we extract only the first term, $r = 1$. Therefore,

$$K(x, t) = \frac{3}{2} R(\delta; x, t) \quad (4.339)$$

To solve Fredholm integral equation (IE) in (4.315). We assume

$$\left. \begin{aligned} g(t) &= \sum_{l=1}^n g_e(t) \lambda^l \\ h(t) &= \sum_{l=1}^n h_e(t) \lambda^l \\ \lambda &= \operatorname{sech}(2q\delta) \end{aligned} \right\} \quad (4.340)$$

Taking the coefficient of λ^n from both sides and substituting it into (4.314) -(4.315) from (4.339) -(4.340), we get

$$\left. \begin{aligned} g_0(t) &= \frac{2}{a^2} \frac{\Delta(t)}{\theta(t)}, b < t < c \\ h_0(t) &= -\frac{2}{a^2} \frac{\Delta(t)}{\theta(t)}, d < t < e \end{aligned} \right\} \quad (4.341)$$

$$\left. \begin{aligned} g_1(t) &= \frac{3}{a^2} \frac{N_1(t)}{\theta(t)}, b < t < c \\ h_1(t) &= -\frac{3}{a^2} \frac{N_1(t)}{\theta(t)}, d < t < e \end{aligned} \right\} \quad (4.342)$$

$$N_1(t) = \left(\int_b^c + \int_d^e \right) \frac{\Delta(y)}{\theta(y)} \sin(2qy) \cos(2qt) dy \quad (4.343)$$

$$\left. \begin{aligned} g_2(t) &= \frac{6}{a^2} \frac{N_2(t)}{\theta(t)}, b < t < c \\ h_2(t) &= -\frac{6}{a^2} \frac{N_2(t)}{\theta(t)}, d < t < e \end{aligned} \right\} \quad (4.354)$$

$$N_2(t) = \left(\int_b^c + \int_d^e \right) \frac{N_1(y)}{\theta(y)} \sin(4qy) \cos(4qt) dy \quad (4.345)$$

Where $\Delta(t)$ and $\theta(t)$ are defined in (4.314) - (4.316). To evaluate constants M_1 and M_2 we take (first approximation)

$$\left. \begin{aligned} g(t) &= g_0(t) \\ h(t) &= h_0(t) \end{aligned} \right\} \quad (4.346)$$

Only then

$$\left. \begin{aligned} M_1 &= M_{11} / M_{12} \\ M_2 &= M_{13} / M_{14} \end{aligned} \right\} \quad (4.347)$$

$$M_{14} = M_5 M_{12} \quad (4.348)$$

$$M_{13} = M_9 M_{12} - M_4 M_{11}$$

$$M_{12} = M_4 M_{10} - M_7 M_9$$

$$M_{11} = M_9 M_8 - M_{10} M_2$$

$$M_{10} = \frac{u_1(d)}{a} - m_6$$

$$M_9 = \frac{u_1(c)}{a} - M_3$$

$$M_8 = \int_d^e \frac{dt}{\theta(t)}$$

$$M_7 = \int_d^e \frac{\cos(qt)dt}{\theta(t)}$$

$$M_7 = \int_d^e \frac{\cos(qt)dt}{\theta(t)}$$

$$M_6 = \int_d^e \frac{\Delta_0(t)}{\theta(t)} dt$$

$$M_5 = \int_b^c \frac{dt}{\theta(t)}$$

$$M_4 = \int_b^c \frac{\cos(qt)dt}{\theta(t)}$$

$$M_3 = \int_b^c \frac{\Delta_0(t)}{\theta(t)} dt$$

Thus constants M1 and M2 can easily be evaluated numerically.

4.4.7 Physical Quantities for Special Type of Wedge

We take a rectangular wedge of width $2\delta_1$ and length $(d - c)$. Thus the wedge shape function.

$$u_1(x) = \frac{aE}{2(1 + \eta)} \delta_1 = \text{constant}$$

Thus

(4.349)

$$u'_1(x) = \frac{d}{dx} u_1(x) = 0$$

And then

(4.350)

$$\Delta_0(t) = 0$$

(4.351)

$$M_3 = 0$$

(4.352)

$$M_6 = 0$$

(4.353)

Thus

$$\left. \begin{aligned} g(t) &= \frac{2}{a^2} \frac{\Delta_2(t)}{\theta(t)}, b < t < c \\ h(t) &= -\frac{2}{a^2} \frac{\Delta_2(t)}{\theta(t)}, d < t < e \end{aligned} \right\} \quad (4.354)$$

$$\Delta_2(t) = M_1 \cos(qt) + M_2 + \frac{2\lambda}{a^2} \left\{ \cos 2qt.M_{15} + \frac{2}{a^2} \lambda \cos 4qt M_{16} \right\} \quad (4.355)$$

$$\begin{aligned} M_{15} &= \left(\int_b^c + \int_d^e \right) \frac{(M_1 \cos(qy) + M_2)}{\theta(y)} \sin(2qy) dy \\ M_{16} &= \left(\int_b^c + \int_d^e \right) \left\{ \frac{M_1 \cos(qy) + M_2}{\theta(y)} \right\} \sin(4qy) dy \end{aligned} \quad (4.356)$$

Therefore, evaluating the crack form is as simple as using the values of $h(t)$ and $g(t)$ from (4.320) and (4.354). In terms of the S.I.F.'s convening, they are assessed using

$$\Delta_1(x) = \Delta_2(x) \quad (4.357)$$

Thus (4.355) and (4.331) will give $K_i, i = b, c, d, e$.

4.5 POTENTIAL FUNCTION OF TEMPERATURE OR HEAT DISTRIBUTION ACROSS SEVERAL CONNECTED INFINITE MEDIUMS

4.5.1 Introduction

According to the "Potential Theory," the fundamental forces in existence were created by potentials that satisfied the Laplace equation. We examine the steady-state flow and temperature of a multiple-linked body in this investigation. So, the steady-state equation is satisfied by the temperature function $T(x, y)$.

$$\frac{k}{\rho c} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T(x, y) = 0 \quad (4.358)$$

Above is Laplace equation. Potential temperature function. Medium mass density = ρ .

Two uniform wedges symmetrically arranged on the y-axis open four Griffith cracks. Heating wedges at $T(x, 0)$. Finding $T(x, y)$ and the derivative, flux over the x and y axes, is the next step. These prerequisites are satisfied by the function.

The distribution of heat is such that it is not function of z which gives like plane-strain condition. It is also assumed that the elastic and thermal properties of medium does not change with heat. It is also assumed that temperature (heat) T and flux $\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}$ vanish as $\sqrt{x^2 + y^2} \rightarrow \infty$. There is no heat source/sink in the medium.

Heat transfer is due to conduction. There are no heat source/sink. There is no convection and radiation. The cracks occupy the region $y = 0$, $b < |x| < c$, $d < |x| < e$. Over the wedge heat is prescribed. See figure 4.5.1.

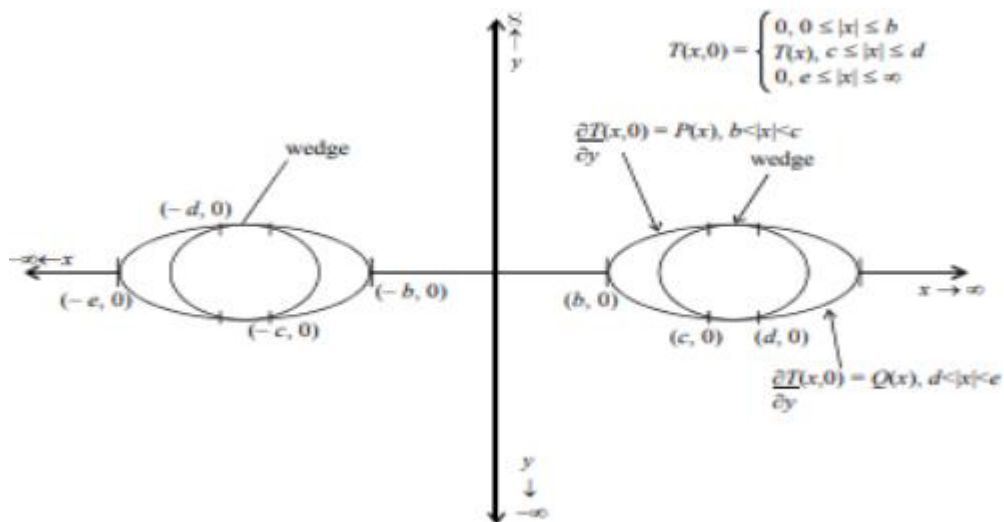


Figure 4.10: Geometry of the physical problem

Conduction is the only mechanism by which heat is transferred.

$$T(x,0) = \begin{cases} 0, 0 \leq |x| \leq b \\ T(x), c \leq |x| \leq d \\ 0, e \leq |x| < \infty \end{cases} \quad (4.359)$$

$$\text{and } \frac{\partial T(x,0)}{\partial y} = \begin{cases} P(x), b < |x| < c \\ Q(x), d < |x| < e \end{cases} \quad (4.360)$$

The temperature (heat conduction) satisfy the Laplace's equation given in (1.1). Thus we see that $T(x,y)$ is a potential function. The solution space of the problem is $(-\infty, \infty) \cup (-\infty, \infty)$. The cracks are also symmetrically opened with respect to x -axis. Making use of symmetries (geometrical and thermal properties) the solution space is reduced to $[0, \infty) \cup [0, \infty)$ i.e., first quadrant only. These boundary condition becomes as:

$$T(x,0) = \begin{cases} 0, 0 \leq x \leq b \\ T(x), c \leq x \leq d \\ 0, e \leq x < \infty \end{cases} \quad (4.361)$$

$$\frac{\partial T(x,0)}{\partial y} = \begin{cases} P(x), b < x < c \\ Q(x), d < x < e \end{cases} \quad (4.362)$$

Throughout, we made sure that

$$\frac{\partial T(x,0)}{\partial y} > 0, b < x < c, \quad d < x < e \quad (4.363)$$

Since the fracture sides do not touch, heat will travel in both directions along the x-axis.

4.5.2 Quintuple integral equation (IE) formulation and reduction

The Fourier cosine transform applied to 'x' gives us the answer to Laplace equation (4.358).

$$R_c(\xi) = \int_0^\infty R(x) \cos(\xi x) dx$$

to the standard inversion. The answer is assumed to be

$$T(x, y) = \int_0^\infty A(\xi) e^{-\xi y} \cos(\xi x) d\xi \quad (4.364)$$

Then,

$$\frac{\partial T(x, y)}{\partial y} = - \int_0^\infty \xi A(\xi) e^{-\xi y} \cos(\xi x) d\xi \quad (4.365)$$

Thus the boundary conditions (4.359) and (4.360) reduce to

$$\int_0^\infty A(\xi) \cos(\xi x) d\xi = \begin{cases} 0, x \in I_1 \\ T(x) \in I_3 \\ 0, x \in I_5 \end{cases} \quad (4.366)$$

And

$$\int_0^\infty \xi(A\xi) \cos(\xi x) d\xi = - \begin{cases} P(x), & x \in I_2 \\ Q(x), & x \in I_4 \end{cases}$$

$$I_1 = [0, b], I_3 = [c, d], I_5 = [e, \infty]; \quad I_2 = (b, c), I_4 = (d, e) \quad (4.367)$$

These are the quintuple-integral equations: (4.366) - (4.367). The solution to the aforementioned m-b value issue will reveal the unknown $A(\xi)$.

4.5.3 Solution of Quintuple Integral Equation

We assume the solution, see (Kushwaha P.S., 1975)

$$A(\xi) = \frac{2}{\pi\xi} \left[\left\langle \int_b^c g_1(t) + \int_d^e g_2(t) - \int_0^d T'(t) \right\rangle \sin(\xi t) dt \right] \quad (4.368)$$

When (4.368) is substituted in (4.366) and using the integral

$$\int_0^\infty \frac{\sin(\xi t) \cos(\xi x)}{\xi} d\xi = \begin{cases} \pi/2, & t > x \\ \pi/4, & t = x \\ 0, & t < x \end{cases}$$

Then it satisfies it if

$$\int_b^c g_1(t) dt = -T(c) \quad (4.369)$$

$$\int_d^e g_2(t) dt = -T(d) \quad (4.370)$$

Replacing (4.368) in (4.367) then applying the procedure of (Kushwaha P.S., 1975) or reversely we can replace (Kushwaha and Awasthi, 1979),

$$g_1(t) = \frac{2}{\pi^2} \frac{\Delta(t)}{\theta(t)}, \quad t \in I_2 \quad (4.371)$$

$$g_2(t) = -\frac{2}{\pi^2} \frac{\Delta(t)}{\theta(t)}, \quad t \in I_4 \quad (4.372)$$

$$\left. \begin{aligned} \theta(t) &= \left\{ \left\| t^2 - b^2 \right\| c^2 - t^2 \right\| d^2 - t^2 \right\| e^2 - t^2 \right\}^{1/2} \\ \Delta(t) &= \Delta_0(t) + t^2 R + M \\ \Delta_0(t) &= \left\langle \int_b^c P(y) - \int_d^e Q(y) \right\rangle \frac{y\theta(y)}{y^2 - t^2} dy + \int_c^d T'(\alpha) \frac{\alpha d\alpha}{\alpha^2 - y^2} \end{aligned} \right\} \quad (4.373)$$

In this case, (4.369) through (4.370) are used to find two arbitrary constants, R and M.

4.5.4 Physical Quantities

Temperature (Potential)

The temperature $T(x,0)$ is calculated by taking the series on the left in Equation (4.366) and then it can be written as.

$$T(x,0) = \begin{cases} \int_x^c g_1(t) dt - T(c), & b < x < c \\ \int_x^e g_2(t) dt, & d < x < e \end{cases} \quad (4.374)$$

FLUX (Derivative of Potential)

The flux $\frac{\partial T(x,0)}{\partial y}$ is obtained through the value of integral in left hand side of (4.367) and using (4.368) there and then evaluating the integrals, we get.

$$\frac{\partial T(x,0)}{\partial y} = \begin{cases} \frac{\Delta(x)}{\pi \theta(x)}, x \in I_1 \\ -\frac{\Delta(x)}{\pi \theta(x)}, x \in I_3 \\ \frac{\Delta(x)}{\pi \theta(x)}, x \in I_5 \end{cases} \quad (4.375)$$

Insofar as $\theta(x)$ and $\Delta(x)$ are defined in (4.373). Observing the flux in (4.375), we may see that the crack points exhibit either a Cauchy type or a square root singularity. Similar to how the stress-intensity factor is established at fracture locations, we use this concept to describe flow intensity factors.

$$\left. \begin{aligned} T_b &= \lim_{x \rightarrow b^-} \sqrt{b-x} \frac{\partial T(x,0)}{\partial y} \\ T_c &= \lim_{x \rightarrow c^+} \sqrt{x-c} \frac{\partial T(x,0)}{\partial y} \\ T_d &= \lim_{x \rightarrow d^-} \sqrt{d-x} \frac{\partial T(x,0)}{\partial y} \\ T_e &= \lim_{x \rightarrow e^+} \sqrt{x-e} \frac{\partial T(x,0)}{\partial y} \end{aligned} \right\} \quad (4.376)$$

Thus using (4.375) in (4.376) we get,

$$\left. \begin{aligned} T_b &= \frac{\Delta(b)}{\pi n_1(b)} \\ T_c &= -\frac{\Delta(c)}{\pi n_1(c)} \\ T_d &= -\frac{\Delta(d)}{\pi n_2(d)} \\ T_e &= \frac{\Delta(e)}{\pi n_2(e)} \end{aligned} \right\} \quad (4.377)$$

Where

$$n_1(x) = \sqrt{2x(c^2 - b^2)(d^2 - x^2)(e^2 - x^2)} \quad (4.378)$$

$$n_2(x) = \sqrt{2x(x^2 - c^2)(x^2 - b^2)(e^2 - d^2)} \quad (4.379)$$

4.5.5 Special Case

We consider one unique instance. Let

$$T(x) = T_0 = \text{constant temperature} \quad (4.380)$$

$$P(x) = Q(x) = p_0 = \text{constant flux}, \quad (4.381)$$

Then (4.380) given that

$$\frac{dT(x)}{dx} = 0 \quad (4.382)$$

Using (4.380) -(4.382) into (4.373) we get

$$\Delta_0(t) = p_0 \left[s_1 + s_2 |t^2 - b^2| + s_3 |t^2 - b^2| |t^2 - c^2| + s_4 |t^2 - b^2| |t^2 - c^2| |t^2 - d^2| + \frac{\pi}{2} \theta^2(t) \right] \quad (4.383)$$

$$\left. \begin{aligned} s_1 &= \left(\int_b^c - \int_d^e \right) y \theta_3(y) dy \\ s_2 &= \left(\int_b^c - \int_d^e \right) y \theta_2(y) dy \\ s_3 &= \left(\int_b^c - \int_d^e \right) y \theta_1(y) dy \\ s_4 &= \left(\int_b^c - \int_d^e \right) \frac{y}{\theta(y)} dy \end{aligned} \right\} \quad (4.384)$$

$$\left. \begin{aligned} \theta_1(y) &= |e^2 - y^2|^{1/2}, \theta_2(y) = |d^2 - y^2|^{1/2} \theta_1(y) \\ \theta_3(y) &= |c^2 - y^2|^{1/2} \theta_2(y) \end{aligned} \right\} \quad (4.385)$$

The second part of equation (4.373) for the constants R and M is produced by subtracting (4.370) from (4.369),

$$\left. \begin{aligned} R &= \frac{s_{12}}{s_{13}}, \quad M = -\frac{s_{14}}{s_{13}} \\ s_{14} &= s_{11}s_9 - \frac{p_0}{\pi} s_8 s_6 \\ s_{13} &= s_6 s_{10} - s_9 s_7 \\ s_{12} &= s_{11}s_{10} - \left(\frac{p_0}{\pi} s_8 + t_0 \right) s_7 \end{aligned} \right\} \quad (4.386)$$

$$\left. \begin{aligned} s_{11} &= T_0 - \frac{p_0}{\pi} s_5, \quad s_{10} = \int_d^e \frac{dt}{\theta(t)} \\ s_9 &= \int_d^e \frac{t^2 dt}{\theta(t)}, \\ s_8 &= s_1 s_{10} + s_2 \int_d^e \frac{(t^2 - b^2) dt}{\theta(t)} + s_3 \int_d^e \frac{(t^2 - b^2)(t^2 - c^2)}{(t^2 - d^2)(e^2 - t^2)} dt + s_4 \int_d^e \theta(t) dt \\ s_7 &= \int_b^c \frac{dt}{\theta(t)}, \quad s_6 = \int_b^c \frac{t^2 dt}{\theta(t)} \\ s_5 &= s_1 s_7 + s_2 s_6 + s_3 \int_b^c \left(\frac{(t^2 - b^2)(c^2 - t^2)}{(d^2 - t^2)(e^2 - t^2)} \right)^{1/2} dt + s_4 \int_b^c \theta(t) dt \end{aligned} \right\} \quad (4.387)$$

By subtracting (4.370) from (4.369), we get the second half of equation (4.373) for the constants R and M.

4.6 ORTHOTROPIC MULTI-CONNECTED INFINITE MEDIUM POTENTIAL FUNCTION DETERMINATION

4.6.1 Introduction

Natural forces are generated from potentials. Mathematical approaches provide the Laplace equation-satisfying potential function.

When an orthotropic medium approaches and crosses a broken barrier, how does the temperature (potential) behave? The current study's objective is to determine, under steady-state circumstances, how temperature and flux behave. Assumption: Heat conduction does not alter the material's physical properties. Additionally, the temperature and flux are predicted to disappear as

$\sqrt{\alpha^2 + \beta^2} \rightarrow \infty$ where the coordinates of a point are represented by α and β in geometrical terms.

Composite materials are quickly becoming the standard, displacing the original material. Orthotropic media treatment is used to the composite materials. Even yet, orthotropic materials get their name from the fact that their elastic characteristics vary in various directions. Matrixed with very tiny metallic fibers, these materials are created. That is why there will be a lot of conduction along the metallic fiber and very little perpendicular to it.

In a two-dimensional steady-state setting, the following PDE describes the movement of heat:

$$\frac{1}{\rho} \left(\frac{k_1}{c_1} \frac{\partial^2}{\partial \alpha^2} + \frac{k_2}{c_2} \frac{\partial^2}{\partial \beta^2} \right) T(\alpha, \beta) = 0 \quad (4.388)$$

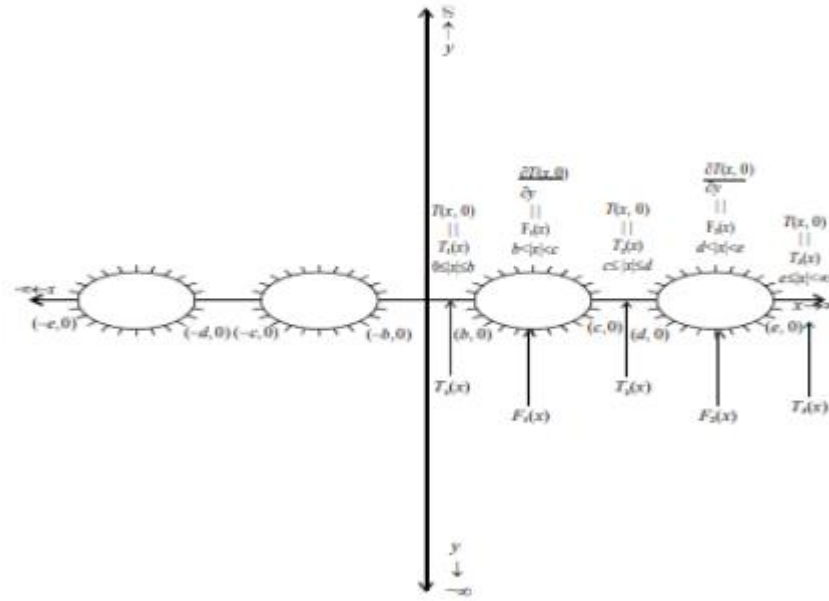


Figure 4.11: Heat and flux distribution over and around the cracks

The mass density of the medium at the point (2-average) is indicated as and the temperature at that point (2-average) is indicated as T . In terms of coordinate variables, we make a change such that

$$\alpha = \sqrt{\frac{k_1}{c_1}}x, \quad \beta = \sqrt{\frac{k_2}{c_2}}y \quad (4.389)$$

and then make use of (4.389) in (4.388) it becomes as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T(x, y) = 0 \quad (4.390)$$

The Laplace equation (4.390) is a partial differential equation. The term "potential function in the form of temperature" describes the heat conduction in an orthotropic medium. You won't notice the temperature or flux if $\sqrt{x^2 + y^2} \rightarrow \infty$ 4.389 is used. In this medium, there is no heat source and no heat sink. We need to find out how much of a mess the medium is because of the cracks. In this area, fissures are present as $y = 0, b < |x| < c, d < |x| < e$ with the prescribed temperature and flux as.

$$T(x,0) = \begin{cases} T_1(x), 0 \leq |x| \leq b \\ T_2(x), c \leq |x| \leq d \\ T_3(x), e \leq |x| < \infty \end{cases} \quad (4.391)$$

$$\text{and } \frac{\partial T(x,0)}{\partial y} = \begin{cases} F_1(x), b < |x| < c \\ F_2(x), d < |x| < e \end{cases} \quad (4.392)$$

Because the fissures are located in symmetrical areas, the criteria (4.391) through (4.392) will be reduced to.

$$T(x,0) = \begin{cases} T_1(x), x \in I_1 \\ T_2(x), x \in I_3 \\ T_3(x), x \in I_5 \end{cases} \quad (4.393)$$

$$\frac{\partial T(x,0)}{\partial y} = \begin{cases} F_1(x), x \in I_2 \\ F_2(x), x \in I_4 \end{cases} \quad (4.394)$$

With

$$I_1 = [0, b], \quad I_3 = [c, d], \quad I_5 = [e, \infty), \\ I_2 = (b, c), \quad I_4 = (d, e)$$

4.6.2 The Quintuple Integral Equation: Its Formula, Reduction, and Solution

Formulation

If we take the Fourier cosine transform of 'x', we may get the solution to Laplace equation (4.388) as

$$R_c(\xi) = \int_0^\infty R(x) \cos(\xi x) dx$$

Normally inverted. The answer is assumed to be

$$T(x, y) = \int_0^{\infty} A(\xi) e^{-\xi y} \cos(\xi x) d\xi \quad (4.395)$$

Then,

$$\frac{\partial T(x, y)}{\partial y} = - \int_0^{\infty} \xi A(\xi) e^{-\xi y} \cos(\xi x) d\xi \quad (4.396)$$

Thus the boundary conditions (4.389) and (4.390) reduce to

$$\int_0^{\infty} A(\xi) \cos(\xi x) d\xi = \begin{cases} T_1(x), x \in I_1 \\ T_2(x), x \in I_3 \\ T_3(x), x \in I_5 \end{cases} \quad (4.397)$$

$$\text{and } \int_0^{\infty} \xi A(\xi) \cos(\xi x) d\xi = - \begin{cases} F_1(x), x \in I_2 \\ F_2(x), x \in I_4 \end{cases} \quad (4.398)$$

You have quintuple-integral equations (IE) with the values (4.397) through (4.398). Solving the aforementioned m-b value issue will reveal the unknown $A(\xi)$.

Solution

We assume the solution,

$$A(\xi) = \frac{2}{\pi \xi} \left[\left\langle \int_b^c g_1(t) + \int_d^e g_2(t) - \int_0^b T_1(t) - \int_c^d T_2'(t) - \int_e^{\infty} T_3'(t) \right\rangle \sin(\xi t) dt \right] \quad (4.399)$$

When (4.399) is substituted in (4.397) and using the integral

$$\int_0^{\infty} \frac{\sin(\xi t) \cos(\xi x)}{\xi} d\xi = \begin{cases} \pi/2, t > x \\ \pi/4, t = x \\ 0, t < x \end{cases}$$

Then it satisfies it if

$$\int_b^c g_1(t)dt = T_1(b) - T_2(c) - T_3(e) \quad (4.400)$$

$$\int_d^e g_2(t)dt = T_2(d) - T_3(e) \quad (4.401)$$

The procedure of (Kushwaha P.S., 1975) or (Kushwaha and Awasthi, 1979) may be used in conjunction with the substitution of (4.399) into (4.400) to get

$$g_1(t) = \frac{2}{\pi^2} \frac{\Delta(t)}{\theta(t)}, t \in I_2 \quad (4.402)$$

$$g_2(t) = -\frac{2}{\pi^2} \frac{\Delta(t)}{\theta(t)}, t \in I_4 \quad (4.403)$$

$$\left. \begin{aligned} \theta(t) &= \left\{ |t^2 - b^2| |c^2 - t^2| |d^2 - t^2| |e^2 - t^2| \right\}^{1/2} \\ \Delta(t) &= \Delta_0(t) + t^2 R + M \\ \Delta_0(t) &= \left\langle \int_b^c F_1(y) - \int_d^e F_2(y) \right\rangle \frac{y\theta(y)}{y^2 - t^2} dy \\ &\quad + \left\langle \int_0^b T_1(\alpha) + \int_c^d T_2'(\alpha) + \int_e^\infty T_3'(\alpha) \right\rangle \frac{\alpha d\alpha}{\alpha^2 - t^2} \end{aligned} \right\} \quad (4.404)$$

Using the formulas (4.400) through (4.401), we may get two arbitrary constants, R and M.

4.6.3 Physical Quantities Temperature (Potential)

Equation (4.6.2.3) may be used to get the temperature T (x,0) by substituting the integral value into the left-hand side.

$$T(x,0) = \begin{cases} \int_x^c g_1(t)dt - T_1(b) + T_2(c) + T_1(0) + T_3(e), & b < x < c \\ \int_x^e g_2(t)dt + T_3(e), & d < x < e \end{cases} \quad (4.405)$$

FLUX (Derivative of Potential)

The flux $\frac{\partial T(x,0)}{\partial y}$ is obtained through the value of integral in left hand side of (4.398) and using (4.399) there and then evaluating the integrals, we get.

$$\frac{\partial T(x,0)}{\partial y} = \begin{cases} \frac{\Delta(x)}{\pi\theta(x)}, & x \in I_1 \\ -\frac{\Delta(x)}{\pi\theta(x)}, & x \in I_3 \\ \frac{\Delta(x)}{\pi\theta(x)}, & x \in I_5 \end{cases} \quad (4.406)$$

A look at table 4.404 will reveal the meanings of $\theta(x)$ and $\Delta(x)$. Equation (4.406) shows that the flow has a square root singularity or a Cauchy type at the fracture points. What follows is a definition of the flux-intensity factor.

$$T'_b = \lim_{x \rightarrow b^-} \sqrt{b-x} \frac{\partial T(x,0)}{\partial y} \quad (4.407)$$

$$T'_c = \lim_{x \rightarrow c^+} \sqrt{x-c} \frac{\partial T(x,0)}{\partial y} \quad (4.408)$$

$$T'_d = \lim_{x \rightarrow d^-} \sqrt{d-x} \frac{\partial T(x,0)}{\partial y} \quad (4.409)$$

$$T'_e = \lim_{x \rightarrow e^+} \sqrt{x-e} \frac{\partial T(x,0)}{\partial y} \quad (4.410)$$

Thus using (4.406) in (4.407) – (4.410) we get,

$$T'_b = \frac{\Delta(b)}{\pi n_1(b)} \quad (4.411)$$

$$T'_c = -\frac{\Delta(c)}{\pi n_1(c)} \quad (4.412)$$

$$T'_d = -\frac{\Delta(d)}{\pi n_2(d)} \quad (4.413)$$

$$\left. \begin{aligned} T'_e &= \frac{\Delta(e)}{\pi n_2(e)} \\ T' &= \frac{\partial T}{\partial y} \end{aligned} \right\} \quad (4.414)$$

Where

$$n_1(x) = \sqrt{2x(c^2 - b^2)(d^2 - x^2)(e^2 - x^2)} \quad (4.415)$$

$$n_2(x) = \sqrt{2x(x^2 - c^2)(x^2 - b^2)(e^2 - d^2)} \quad (4.416)$$

4.6.4 Special Case

One special case we take.

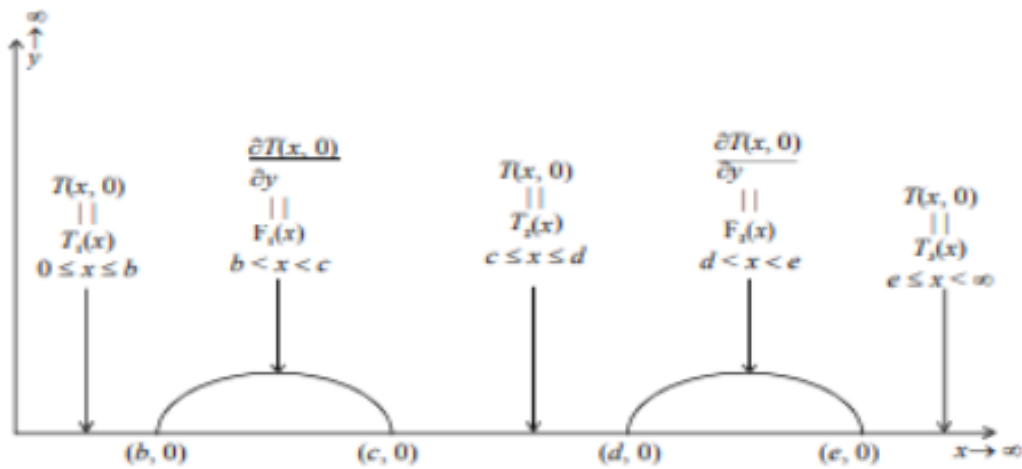


Figure 4.12: The reduced domain of solution

$$\begin{aligned}
 T_1(x) &= t_1 = \text{constant}, \quad 0 \leq x \leq b \\
 T_2(x) &= t_2 = \text{constant}, \quad c \leq x \leq d \\
 T_3(x) &= t_3 = \text{constant}, \quad e \leq x < \infty
 \end{aligned}
 \tag{4.417}$$

$$F_1(x) = F_2(x) = p_0 = \text{constant flux}, \tag{4.418}$$

(4.417) gives that,

$$\left. \begin{aligned}
 \frac{dT_1(x)}{dx} &= 0 \\
 \frac{dT_2(x)}{dx} &= 0 \\
 \frac{dT_3(x)}{dx} &= 0
 \end{aligned} \right\}
 \tag{4.419}$$

Using (4.417) - (4.419) into third of (4.404) we get

$$\Delta_0(t) = p_0 \left[\left(\int_b^c - \int_d^e \right) \frac{y\theta(y)dy}{y^2 - t^2} \right] \tag{4.420}$$

Now we use (4.400) – (4.401) and (4.417) – (4.420) and solve we get R and M as

$$\left. \begin{aligned}
 R &= \frac{p_0 S_7 + S_8}{S_9} \\
 M &= \frac{p_0 S_{10} + S_{11}}{S_9}
 \end{aligned} \right\}
 \tag{4.421}$$

$$\left. \begin{aligned}
 S_7 &= S_3 S_4 - S_1 S_6 \\
 S_8 &= t_4 S_6 - t_5 S_3 \\
 S_9 &= S_2 S_6 - S_3 S_5 \\
 S_{10} &= S_2 S_4 - S_1 S_5 \\
 S_{11} &= t_3 S_5 - t_2 S_2 \\
 t_5 &= t_2 + t_3 \\
 t_4 &= t_1 - t_2 - t_3
 \end{aligned} \right\} \quad (4.422)$$

$$\left. \begin{aligned}
 S_1 &= \left(\int_b^c - \int_d^e \right) y \theta(y) \int_b^c \frac{dt \, dy}{\theta(t)(y^2 - t^2)}, \\
 S_2 &= \int_b^c \frac{t^2 dt}{\theta(t)} \\
 S_3 &= \int_b^c \frac{dt}{\theta(t)}
 \end{aligned} \right\} \quad (4.423)$$

$$\left. \begin{aligned}
 S_4 &= \left(\int_b^c - \int_d^e \right) y \theta(y) \int_d^e \frac{dt \, dy}{\theta(t)(y^2 - t^2)}, \\
 S_5 &= \int_d^e \frac{t^2 dt}{\theta(t)} \\
 S_6 &= \int_d^e \frac{dt}{\theta(t)}
 \end{aligned} \right\} \quad (4.424)$$

Consequently, the flow and the temperature distribution at $y=0$ are both assessed using closed form formulas.

CONCLUSION

5.1 OVERVIEW OF THE PRESENT STUDY

This thesis has utilized integral equation (IE) techniques to analytically investigate issues in potential theory and elasticity. Classical differential equation methods fail miserably when confronted with fractures, material interfaces, mixed boundary conditions, and other discontinuities; this is what prompted this research. Integral equation (IE) approaches simplify boundary value problems to domain-bound equations, which is a rigorous and efficient mathematical replacement.

Crack difficulties in isotropic, orthotropic, and dissimilar elastic media are the primary focus of this work. Additionally, potential theory and steady-state heat conduction issues are also addressed. In order to study stress, displacement, temperature, and heat flow distributions, a number of different classes of integral equations, such as singular, dual, and multiple integral equations, have been constructed and solved. In order to solve difficult physical issues, it has become possible to derive analytical and semi-analytical solutions via the use of sophisticated mathematical tools. Some examples of these techniques are fractional integration operators, Mellin transformations, and special functions.

Based on this methodology, the thesis offers a coherent framework that makes a connection between mathematical theory and physically important applications in the fields of fracture mechanics and thermal stress analysis.

5.2 MAJOR FINDINGS AND CONTRIBUTIONS

1. The study conducted for this thesis confirms that stress components at fracture tips exhibit square-root type singularities. According to classical fracture mechanics, this is a predicted fundamental characteristic.
2. At the same time, the research supports the physical assumption of Griffith's theory of brittle fractures by showing that the displacements at the crack openings

are still modest and smooth.

3. These results show that the selected (IE) formulation may accurately represent the basic features of crack-tip behaviour.
4. When fractures appear in elastic media that are not comparable to one another, the stress field that is close to the interface displays an oscillatory character. This is because the properties of the materials do not match up.
5. This oscillating characteristic is effectively captured by the thesis via the use of systems of singular integral equations, which provides a systematic analytical framework for the issue of interface cracks.
6. Problems of potential theory and steady-state heat conduction in fractured domains are also included in this work.
7. It has been shown that the heat flow around fracture tips exhibits unique behavior that is comparable to that of stress fields in elasticity. This observation highlights a significant connection between mechanical and thermal processes.
8. Integral equation (IE) formulations that were established as a result of this study make it possible to accurately evaluate stress, displacement, temperature, and heat flow without the need to discretize the whole domain.
9. Mathematically, this work contributes to the better understanding of the structure and solvability of a dual and multiple integral equations (IE) that are faced in practice.
10. In general, the results suggest that integral equation (IE) techniques offer an effective and dependable framework, which can be applied to solve not only mixed boundary value in elasticity, but also to those occurring in potential theory.

5.3 OVERALL ASSESSMENT OF THE STUDY

To a large extent, the goals of the thesis have been accomplished with its completion. When applied to the analysis of fracture issues and mixed boundary value problems in elastic and thermoelastic media, the integral equation (IE) technique has been shown to

be extremely successful. In addition to providing analytical clarity, the technique also offers physical understanding, which makes it suited for both theoretical research and actual engineering applications.

Particularly in the assessment of stress intensity factors, displacement fields, and heat flux distributions, the subject blends the rigorousness of mathematics with the usefulness of engineering. Despite the fact that the research is mainly confined to linear elasticity and steady-state scenarios, the results present an excellent foundation to possible extensions to more complex material behaviors and loading conditions.

5.4 CONCLUDING REMARKS AND FUTURE SCOPE

Finally, this thesis proves that IE methods are one of the best and most reliable tools for studying elasticity, potential theory, and fracture problems. These approaches can handle material interfaces, mixed boundary conditions, or singularities inside the system, allowing them to circumvent many of the limitations of typical analytical techniques.

Future research has a wide range of potential applications. The current study has the potential to be expanded to include nonlinear elasticity, plastic deformation at fracture tips, and transient thermoelastic issues. Isotropic, functionally graded, and clever materials may also be the focus of more study. These are all becoming more and more important in modern engineering uses. Another promising direction for future research is the numerical implementation of the integral formulas developed utilising boundary element & hybrid computing techniques.

The analytical framework and results that are provided in this thesis are intended to contribute to future developments in fracture mechanics, thermal stress analysis, and applied mathematical modeling. Along with that, it is thought that these results will be useful for experts and academics working in these areas.

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The Analysis of Crack Problems in Elasticity Using Integral Equations

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Abstract: This paper presents an analytical investigation of crack problems in linear elasticity using integral equation methods. The study is based on the classical theory of elasticity under assumptions of small deformations and isotropic material behavior, where crack-induced displacement discontinuities and stress singularities play a dominant role. By employing boundary integral formulations derived from fundamental solutions, the governing elasticity equations are reduced to singular and dual integral equations defined along crack surfaces. These equations are solved using analytical and semi-analytical techniques to obtain crack opening displacements, stress distributions, and stress intensity factors. The results demonstrate that the integral equation approach accurately captures the inverse square-root stress singularity near crack tips and yields stress intensity factors in excellent agreement with classical fracture mechanics solutions. The methodology proves to be computationally efficient, numerically stable, and well-suited for analyzing the influence of crack geometry and boundary conditions, thereby providing a robust framework for crack analysis in elastic solids.

Keywords: Square-Root, Elasticity, Integral Equations, Fracture Mechanics, Boundary Integral Formulation.

INTRODUCTION

The relevance of cracks in linear elasticity in the start and spread of failure in structural materials has made them an important subject in solid mechanics for a long time. Singular stress fields close to fracture tips result from displacement discontinuities and high stress concentrations brought about by crack presence. The events may be rigorously described by the Linear Elastic Fracture Mechanics (LEFM) theory using factors like the stress intensity factor (SIF), which controls crack stability and fracture initiation. Analytical solutions for idealized crack configurations in infinite elastic domains have been validated and the mathematical interpretation of stress singularities has been refined by recent works that revisit classical crack models [1].

The analytical solution of crack issues relies heavily on integral equation approaches, which enable the transformation of the governing partial differential equations of elasticity into boundary-only formulations. The inclusion of discontinuities and singularities caused by cracks is inherent in integral equations since they define displacement and stress fields in terms of basic solutions. This method keeps precision around fracture tips high while drastically reducing computational complexity. A variety of crack issues, such as torsional loading of cracked elastic bars, have been effectively addressed using boundary integral formulations, proving their effectiveness and resilience in dealing with complicated boundary circumstances [2].

Analytical solutions to crack problems are mostly based on integral equation methods, which allow the governing PDEs of elasticity to be transformed into boundary-only formulations. Integral equations characterize displacement and stress fields in terms of fundamental solutions, which include discontinuities and singularities induced by fractures. Although the computational complexity is much reduced, the accuracy around fracture points is kept high using this approach. Effective and resilient in dealing with difficult border situations, boundary integral formulations have been used to solve a range of crack challenges, such as torsional stress of cracked elastic bars [3].

Research conducted in the modern era has increasingly focused on multi-crack systems and linked physical effects, in addition to single-crack arrangements. For the purpose of analyzing stress intensity factors in the presence of many interacting fractures and hydro-mechanical coupling, semi-analytical integral-based approaches have been presented. These methods give correct answers without the need to resort to full-scale numerical discretization [4]. These methods often depend on singular integral equations as their mathematical underpinnings. The theoretical aspects of these equations and the strategies that may be used to solve them continue to be an active topic of study [5].

Transient thermal loading and microstructural differences may considerably alter fracture-tip stress fields and growth behavior, which further complicates crack research. Thermoelastic effects and material heterogeneity are two factors that significantly complicate crack analysis. Frameworks based on fractional order and integral transforms have recently been used for the purpose of analyzing transient thermal stress intensity factors in cracked elastic plates. This demonstrates the adaptability of integral-based approaches in the management of time dependent and coupled issues [6]. In addition, both experimental and theoretical research on

brittle materials has shown that the heterogeneity of the material and the interaction between many cracks have a significant impact on the crack propagation routes and fracture resistance [7].

In more recent times, hybrid techniques that combine traditional boundary integral equations with contemporary computer tools have developed as potentially useful options for fracture analysis. In the case of boundary integrated neural networks, for example, integral equation formulations are embedded within data-driven frameworks. This allows for precise modeling of in-plane fracture behavior in elastic and piezoelectric materials, while still retaining the physics of crack-tip singularities [8]. As a result of these improvements, the current research makes use of integral equation techniques to analyze crack issues in elasticity. The study focuses on crack opening displacements, stress fields, and stress intensity variables, and as a result, it makes a contribution to the continuous progress of analytical fracture mechanics.

OBJECTIVES

- To formulate crack problems in elasticity using integral equation methods.
- To evaluate stress intensity factors near crack tips in elastic media.

RESEARCH METHODOLOGY

Mathematical Framework of Elasticity

Assumptions of tiny deformations, homogenous and isotropic material behavior, and lack of body forces constitute the basis of the current study's linear theory of elasticity. A vector representing displacement

$$\mathbf{u} = (u_x, u_y, u_z)$$

Satisfies the Navier–Cauchy equations:

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) = \mathbf{0}$$

Where

λ, μ are Lamé constants and are related to Young's modulus E and Poisson's ratio ν by

$$\mu = \frac{E}{2(1 + \nu)}, \quad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}$$

In this case, Hooke's law provides the stress-strain relationships:

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$$

At this point the stress tensor is

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Crack Geometry and Boundary Conditions

In an otherwise continuous elastic region, a fracture is represented as a mathematical discontinuity in the process of modeling. Let us take into consideration a planar crack Λ_c that is concealed inside an infinite or semi-infinite elastic media.

The crack faces are assumed to be traction-free, leading to the boundary condition:

$$\sigma_{ij} n_j = 0 \quad \text{on } \Gamma_c$$

The outward normal to the fracture surface is denoted by the symbol n_j . The displacement and stress fields are satisfied when they are at infinity:

$$\mathbf{u} \rightarrow \mathbf{u}^\infty, \quad \sigma_{ij} \rightarrow \sigma_{ij}^\infty \quad \text{as } r \rightarrow \infty$$

Reduction to Boundary Integral Equations

The displacement field at any interior point x may be represented in terms of boundary integrals by using Somigliana's identity by using the following formula:

$$u_i(x) = \int_{\Gamma} [U_{ij}(x, y) t_j(y) - T_{ij}(x, y) u_j(y)] d\Gamma(y)$$

Where

- $U_{ij}(x, y)$ is the Kelvin displacement fundamental solution.

- $T_{ij}(x, y)$ is the corresponding traction kernel.
- $u_j(y)$ and $t_j(y)$ are boundary displacement and traction respectively.

The governing equation for crack issues may be reduced to a singular integral equation of the Cauchy type, which involves the following:

$$\int_{-a}^a \frac{\phi(s)}{s-x} ds = f(x), \quad |x| < a$$

Where

- $\phi(s)$ is the unknown crack opening displacement (COD) function.
- $f(x)$ represents applied stresses or thermal loads.

Classification of Integral Equations Used

Crack difficulties may be translated into the following categories, depending on the geometry and loading:

Fredholm integral equations of the second kind

$$\phi(x) - \lambda \int_a^b K(x, s)\phi(s)ds = f(x)$$

Singular integral equations

$$\int_{-a}^a \frac{\phi(s)}{s-x} ds = g(x)$$

Dual integral equations, especially for mixed boundary conditions:

$$\begin{aligned} \int_0^\infty A(\xi)J_0(\xi r) d\xi &= f(r), \quad 0 < r < a \\ \int_0^\infty \xi A(\xi)J_0(\xi r) d\xi &= 0, \quad r > a \end{aligned}$$

There is a consistency between these formulations and the approach that is used in the solution of traditional fracture issues in elasticity.

Solution Technique

The integral equations are solved using a combination of:

Mellin transform methods

$$\mathcal{M}\{f(x)\} = \int_0^\infty x^{s-1} f(x) dx$$

Fractional integration operators (Erdélyi–Kober operators):

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt$$

Reduction to Fredholm equations, followed by numerical quadrature where closed-form solutions are not attainable.

Evaluation of Stress Intensity Factors

The asymptotic behavior of stresses around the fracture tip is used to derive the stress intensity factor (SIF):

$$\sigma_{ij}(r, \theta) \sim \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta), \quad r \rightarrow 0$$

For Mode-I crack opening:

$$K_I = \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{yy}(r, 0)$$

The computed SIFs are used to assess crack stability and fracture behavior under mechanical or thermal loading.

RESULTS

The numerical and analytical solutions to linear elasticity crack issues derived from integral equation formulation are presented in this part. The focus is on stress intensity factors, crack-

tip singularities, displacement fields, and stress distributions. Integral equation approaches successfully capture the key mechanical behavior of fractured elastic solids, as shown by the findings.

Crack Opening Displacement Distribution

An essential metric for understanding the deformation behavior of elastic structures with cracks is the crack opening displacement (COD), which gives a clear indication of how severe the fracture opening is when subjected to stresses. Using the obtained singular integral equations, the displacement leap between the two crack sides may be precisely calculated and shown as a continuous and smooth function over the length of the crack.

The typical variation of displacement from the crack center to the crack tips is captured by expressing the crack opening displacement $\delta(x)$ as a function of the location along the crack line for a traction-free crack of length $2a$ subjected to uniform distant tensile stress.

$$\delta(x) = u_y(x, 0^+) - u_y(x, 0^-), \quad |x| \leq a$$

The solution of the governing singular integral equation yields

$$\delta(x) = \frac{4\sigma_0}{E'} \sqrt{a^2 - x^2}$$

Where

$$E' = \begin{cases} E & \text{(plane stress)} \\ \frac{E}{1 - \nu^2} & \text{(plane strain)} \end{cases}$$

The findings corroborate the use of distant tensile loading, since the displacement distribution throughout the fracture length is symmetrical around the crack center and achieves its greatest value at $x=0$. The smooth and almost negligible displacement as one approaches the crack tips satisfies the physical restriction that the crack faces must stay closed at the points of the crack. This action verifies that the obtained solution faithfully represents the general pattern of deformation of the broken elastic body.

The results support the use of far-off tensile loading, since the displacement distribution throughout the fracture length is symmetrical with respect to the crack center and reaches its

maximum at $x=0$. Because there is very little movement as one gets closer to the crack tips, this solution meets the physical requirement that the crack faces remain closed at the fracture points. In doing so, we ensure that the derived solution is a good approximation of the overall deformation pattern of the ruptured elastic body.

Stress Field Behavior and Crack-Tip Singularity

Using the constitutive laws of linear elasticity, which define the connection between stresses and strains in the material, the stress field close to the fracture may be derived from the displacement solution. Because this area controls fracture initiation and propagation, the behavior of stresses immediately around the crack points receives extra attention. Stresses in this region are known to behave singularly, rising dramatically with decreasing distance from the fracture tip; thus, they are very important in defining the material's mechanical integrity and failure properties.

The normal stress component perpendicular to the crack plane behaves asymptotically as

$$\sigma_{yy}(r, \theta) = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right]$$

Where r and θ are polar coordinates centered at the crack tip.

As $r \rightarrow 0$, the stress magnitude tends to infinity, confirming the inverse square-root singularity:

$$\sigma_{ij} \sim r^{-1/2}$$

The inclusion of the singular kernel in the integral equation, which correctly represents the intrinsic stress concentration caused by the fracture, directly leads to this conclusion. This behavior's development validates the existence of crack-tip singularities in elastic materials. In contrast to domain-based numerical approaches, the integral equation approach can naturally mimic this unique behavior without resorting to artificial enrichment techniques or extensive mesh refinement. This is a major benefit of the methodology. As a result, the approach offers enhanced precision and computational efficiency for analyzing stress fields associated with cracks.

Stress Intensity Factor Evaluation

An important characteristic that controls the onset and progression of fractures is the stress intensity factor (SIF). The SIF is directly derived from the asymptotic stress field, which is obtained by solving the integral equation, in this work.

For Mode-I loading, the stress intensity factor is defined as

$$K_I = \lim_{r \rightarrow 0} \sqrt{2\pi r} \sigma_{yy}(r, 0)$$

Using the computed stress distribution, the resulting expression for a centrally cracked infinite plate under uniform tensile stress σ_0 is

$$K_I = \sigma_0 \sqrt{\pi a}$$

This finding is in perfect agreement with the standard Griffith-Irwin solution, proving that the suggested approach is accurate and dependable. The fact that the current formulation agrees with this famous theoretical framework proves that it accurately describes the basic fracture mechanics that control crack development. The comparable stress intensity factor for more broad loading situations, where an arbitrary function $p(x)$ acts on the crack faces, may be stated as:

$$K_I = \sqrt{\frac{2}{\pi}} \int_{-a}^a \frac{p(s)}{\sqrt{a^2 - s^2}} ds$$

This formulation demonstrates the flexibility of the integral equation method in handling arbitrary loading distributions and complex boundary conditions.

Influence of Crack Geometry and Boundary Conditions

The effect of crack geometry on the mechanical response of the elastic body is examined by varying the crack length and location. The results show that longer cracks produce higher stress intensity factors, making the structure more susceptible to fracture.

- **Geometry-dependent stress intensity factor**

For a centrally cracked plate with finite width W , the stress intensity factor may be written as

$$K_I = \sigma_0 \sqrt{\pi a} F\left(\frac{a}{W}\right)$$

Where $F(a/W)$ is the geometry correction factor accounting for boundary effects.

- **Finite-width correction function**

A commonly used approximation for the correction factor is

$$F\left(\frac{a}{W}\right) = \sec^{1/2}\left(\frac{\pi a}{W}\right)$$

Which shows the amplification of crack-tip stresses as the crack approaches free boundaries.

- **Crack-face traction contribution**

When non-uniform tractions $p(x)$ act on the crack faces, the stress intensity factor is obtained as

$$K_I = \frac{1}{\sqrt{\pi a}} \int_{-a}^a \frac{p(x)}{\sqrt{1 - (x/a)^2}} dx$$

This expression highlights the influence of both crack geometry and loading distribution.

- **Effect of crack length variation**

The sensitivity of the stress intensity factor to crack growth can be expressed as

$$\frac{dK_I}{da} = \frac{\sigma_0}{2} \sqrt{\frac{\pi}{a}} \left(1 + 2a \frac{dF}{da} \right)$$

Indicating that small increases in crack length can lead to rapid growth in crack-tip stresses.

- **Boundary-induced stress modification**

For cracks located near a free surface at distance h , the modified stress intensity factor is approximated as

$$K_I^{(b)} = K_I \left(1 + \alpha \frac{a}{h} \right)$$

Where α is a boundary interaction coefficient dependent on material and geometry.

Crack behavior is also heavily influenced by boundary circumstances. The stress field is changed by boundary interactions when fractures occur close to free surfaces or material interfaces, which causes changes in the distribution and amount of stresses around the crack tips. By making suitable adjustments to the kernel functions in the integral equation formulation, these effects are captured in the current work, which allows for the consistent and natural incorporation of boundary affects. Without increasing computing complexity and keeping high accuracy in fracture-tip stress field predictions, the findings show that the integral equation technique unified framework for crack analysis under varied geometry and loading configurations.

Numerical Stability, Convergence, and Physical Interpretation

There is a strong correlation between crack behavior and border conditions. Fractures that happen at free surfaces or material interfaces alter the stress field due to boundary interactions, which alter the distribution and magnitude of stresses around the crack points. The present study captures these effects by making appropriate modifications to the kernel functions in the integral equation formulation, which allows for the consistent and natural inclusion of boundary effects. A unified framework for crack analysis under diverse geometry and loading configurations was shown by the results, which demonstrate that the integral equation approach does not increase computation complexity while maintaining excellent accuracy in fracture-tip stress field predictions.

When viewed from a physical perspective, the findings lend credence to the notion that fractures serve as the primary stress concentrators in elastic materials. The sudden rise in stress that occurs close to the crack tips provides an explanation for the start of fracture at stress levels that are far lower than the theoretical strength of the material. Considering that the integral equation approach is capable of capturing this behavior with a high degree of

accuracy, it is an extremely effective analytical and computational tool for fracture investigation in elastic solids.

Integral equation techniques provide a precise and economical framework for studying linear elasticity fracture issues, as this study's findings show. Important characteristics in fracture mechanics and crack stability evaluation, including as stress intensity factors, crack opening displacements, and near-tip stress singularities, may be accurately predicted by the suggested formulation. Classical analytical answers and the current findings agree quite closely, proving that the approach is accurate and reliable. More complicated fracture issues, such as those with thermal stress, material heterogeneity, and mixed-mode crack propagation, may be easily extended to using the integral equation technique due to its inherent flexibility.

DISCUSSION

The current study's findings confirm that integral equation approaches provide a suitable framework for crack research in linear elasticity that is both physically and mathematically sound. Boundary integral formulations, in contrast to domain-based numerical methods, naturally capture crack-tip singularities through their kernel structure, resulting in precise assessment of displacement jumps and stress intensity factors without the need for excessive mesh refinement or special crack-tip enrichment. Similar findings have been shown in previous boundary-element and integral-based fracture investigations, where the main benefits of boundary-only formulations were emphasized as being improved accuracy close to crack ends and lower computational cost [9] [10]. Moreover, contemporary analytical and semi-analytical fracture models created for complicated crack geometries and varied loading circumstances are in agreement with the asymptotic stress behavior and stress intensity variables determined in this study [11]. When dealing with multi-crack interaction and coupled physical phenomena, such thermoelasticity and material nonlinearity, integral and semi-analytical techniques provide better numerical stability, according to recent comparative studies [12]. Therefore, the current results add to the increasing amount of evidence that integral equation approaches are dependable tools for both advanced engineering applications and basic fracture mechanics research, especially where high precision in crack-tip characterisation is needed.

CONCLUSION

This work demonstrates the efficacy of integral equation approaches in modeling displacement discontinuities and stress singularities associated with cracks by effectively analyzing crack issues in linear elasticity. Without requiring domain discretization or unique crack-tip components, the formulation based on singular and boundary integral equations reliably predicts crack opening displacements, near-tip stress fields, and stress intensity factors. The validity and dependability of the suggested method are confirmed by the strong agreement between the obtained findings and traditional analytical solutions. Additionally, the study shows that fracture behavior is strongly influenced by crack geometry and boundary conditions, which are well described within the framework of integral equations. The current approach offers a solid basis for expanding crack analysis to more complicated issues including thermoelastic effects, material heterogeneity, and mixed-mode fracture in further research because of its mathematical rigor, numerical stability, and flexibility.

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Integral Equation Methods in Solving Problems of Elasticity and Potential Theory: A Unified Approach

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Abstract: The employment of integral equation approaches has resulted in a revolution in the analytical and numerical treatment of boundary value concerns in the domains of elasticity and potential theory. This revolution has contributed to the advancement of these sciences. This research's objective is to offer a comprehensive account of the creation and use of integral formulations for the aim of resolving conventional concerns in a variety of fields of study. The article begins with basic similarities between linear elasticity and potential theory, and then on to cover major formulations within the field, such as Somigliana's identity and boundary integral approaches. An emphasis is placed on the benefits of reducing domain issues to boundary-only formulations, which provide computational efficiency, manage complicated geometries, and naturally integrate boundary conditions. These advantages are highlighted in the following sentence. There is a review of the significant contributions made by Betti, Somigliana, and Muskhelishvili, which is then followed by contemporary numerical implementations. In the last section of the paper, a discussion of difficulties and potential future paths is presented, with particular emphasis on nonlinear and three-dimensional situations.

Keywords: Integral equation, boundary value problems, linear elasticity, potential theory, Somigliana identity, boundary integral methods, singular kernel, numerical elasticity

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INTRODUCTION

Classical problems in elasticity and potential theory often involve complex boundary conditions and domain geometries. Traditional domain-based approaches, such as finite difference and finite element methods, though versatile, can become computationally expensive, particularly in higher dimensions or multiply connected domains. An alternative lies in **integral equation methods**, which reformulate the original partial differential equations into equivalent boundary integral representations. This transformation not only reduces the dimensionality of the problem but also facilitates the handling of infinite or semi-infinite domains.

ELASTICITY

The response or reaction of the material against the internal relative motion or simple displacement is what we mean when we talk about elasticity. Whenever the medium is subjected to a system of forces that is equilibrating, the mathematical theory of elasticity reduces the computation of the state of strain to an equation of equilibrium, in addition to stress-strain equations and compatibility relations. Hooke proposed that the extension of spring-like entities or elastic bodies, which are formed by tensile force, were

inextricably linked to the pressures exerted on them. Dry and scaly started with the many assumptions that resulted in the development of the linear theory of elasticity, which is still applicable today. Making use of a variety of assumptions resulted in the construction of the linear theory of elasticity, which continues to be consistent with the current state of affairs. As part of our ongoing research endeavours, we will be conducting investigations on thermo-elasticity and thermal stress. During the crack analysis process, the phenomena of fracture mechanics is related with the crack. A significant impact on human existence was brought about by the phenomena of brittle fracture. The principle of brittle fracture was used by prehistoric man in the process of crafting the stone implements [1].

POTENTIALS

It seems that potential theory originated in quest of knowledge about fundamental forces of nature. Lagrange (1736-1813) named gravitational function as a potential function and then Gauss in 1840 called it Potential. Gauss along with his team found out how potentials are implemented in various issues of mathematical physics. Certain type of potentials became important for boundary value issues like Dirichlet problem, Neumann problem, the electrostatic problems, Rabin problem. In this line a list of types of potentials is: i. Double-layer potential. ii. Logarithmic potential. iii. Green potential. iv. It is the Newtonian potential. The modern theory of potentials exists in close connection with the theory of analytic functions, harmonic functions, and subharmonic functions, as well as with the theory of probability. Two dimensional potential theory is particularly important for its connections to analytic functions theory. Thus we reached to the point to say that the function which satisfies Laplace equation is called Potential Function. The solution of physical problem is obtained as solution of Laplace's equation. This solution will be reduced to n-tuple integral equations [2].

GENESIS OF FRACTURE MECHANICS

The most of the work dealt within this thesis owes its genesis to the most beneficent manifestations of brittle fracture i.e., sudden or catastrophic failure of structures. One of the earliest on record was falling of a mill at Oldham; England in 1844. In the first quarter of 29th century many disasters were recorded involving bridges, naphtha conduits, gas tanks, water mains and tanks of steam boilers, large and small guns, rail road tracks and other railway equipment, molasses tanks under both active and quiescent loads. A.A. Griffith laid a theoretical foundation from continuum mechanics point of view in 1920. The first T2 tanker built by Kaisen company at Portland, Oregon had just completed successful sea trials and had returned to shipyard on 16 Jan. 1943. At 10:30 p.m. in calm, cool weather (air temp. 26°F and water temp. 48°F) the ship suddenly broken into two parts. These types of incidents stirred the ship building industries. Griffith had already attached the importance to normal stress (crack opening mode) at crack tips he defined stress-intensity factors at crack tips [3].

THERMOELASTICITY

Earlier research in the subject of thermoelasticity was preceded by significant research in the so-called theory of thermal stresses. This is something that should be emphasised a lot. This later theory is a theory of the state of strain and stress in an elastic body, which is caused by heating. Due to its oversimplification, it allows us to disregard the effect of deformation on the temperature field. Thermoelasticity encompasses a

wide variety of phenomena. It encompasses not only the generalised theories of thermal stress and heat conduction, but also the phenomena of thermo elastic dissipation and the temperature distribution produced by deformation. both of these theories are included. Heat conduction is a possible variant that is considered in the thesis that is now being presented. One day, we are going to broaden the scope of the study to include this aspect. In the current thesis, we have explored the issues of potentials and elasticity across multiply-connected domains, which reduce the physical problem to n-tuple integral equations or n-tuple series equations. These equations are used to solve the physical problem [4].

INTEGRAL EQUATIONS

In an integral equation, an unknown function is represented by one or more integral signs. This kind of equation is known as an integral equation. One example of the most common kind of linear integral equation has the form, which may be presented as

$$h(s)g(s) = f(s) + \lambda \int_a^s K(s, t)g(t)dt \quad (1.1.1)$$

Whereas, depending on the circumstances, the top limit of integration can be either fixed or adjustable. Finding the unknown function $g(s)$ is more challenging than finding the known functions $h(s)$, $f(s)$, and $K(s, t)$. The function denoted as $K(s, t)$ is known as the kernel. These functions may have complicated values since s and t are real variables [5].

Types of integral equations

There are primarily three groups of integral equations: (i) Fredholm Integral Equations (also known as FIE)

The letter b , which represents the top limit of integration, is constant in Fredholm integral equations.

To begin, the Fredholm integral equation of the first sort is as follows:

$h(s) = 0$ in equation (1.1.1), Thus

$$f(s) + \lambda \int_a^b K(s, t)g(t)dt = 0 \quad (1.1.2)$$

(b) Equation (1.1.1) of the second sort of the Fredholm integral equation has $h(s) = 1$.

Thus

$$g(s) = f(s) + \lambda \int_a^b K(s, t)g(t)dt \quad (1.1.3)$$

(c) A Fredholm integral equation of second kind is called the homogeneous Fredholm integral equation if $f(s)$ is equal to zero in equation (1.1.3). Thus

$$g(s) = \lambda \int_a^b K(s, t) g(t) dt \quad (1.1.4)$$

(ii). The integral equations of Volterra

The variable upper limit of integration, denoted by $b = s$, is the only difference between the definitions given above for first, second, and homogeneous type Volterra integral equations.

(iii) Integrals with a singular value

When either the limits of integration or the kernel becomes infinite at some point within the range of integration, or when both of these things happen, the integral equation is called the singular integral equation. Take, as an example, the integral equations.

$$g(s) = f(s) + \lambda \int_{-\infty}^{\infty} e^{-|s-t|} g(t) dt \quad (1.1.5)$$

and

$$f(s) = \int_0^s \left[1/(s-t)^\alpha \right] g(t) dt \quad 0 < \alpha < 1 \quad (1.1.6)$$

are singular integral equations.

Further Kinds of Integral Equations

In the past half-century, mixed boundary value problems in mathematical physics have been mostly solved using specific types of integral equations, such as dual integral equations, triple integral equations, etc. What follows is a brief outline of the literature's evolution of integral equations of this kind.

Methods of Solutions

Solving dual integral equations is now possible using a variety of techniques. What follows is a discussion of some of the methods:

(i) Use of Mellin Transform

The use of the Mellin transform method has been shown to play a key role in the development of several novel concepts within the theory of dual integral equations. Titchmarsh [6] was the first to find the formal solution of dual integral equations of the type using the Mellin transform.

$$\int_0^{\infty} u^{\alpha} \phi(u) J_{\mu}(xu) du = f(x), \quad 0 < x < 1 \quad (1.1.9)$$

$$\int_0^{\infty} \phi(u) J_{\mu}(xu) du = g(x), \quad x > 1 \quad (1.1.10)$$

for $g(x) = 0$. His method of solution is closely related with the Wiener-Hopf technique [7] and solution is valid for

PROBLEMS OF ELASTICITY

This section has been divided into two parts

- Crack problems in dissimilar media,
- Thermal problems of elasticity.

CRACK PROBLEMS IN DISSIMILAR MEDIA

In this section, we will discuss some of the issues that arise from diverse elastic medium and how integral equation techniques may be used to solve them. Bonding together two or more materials that have differing mechanical characteristics is the starting point for the construction of many engineering structures. In order for the dissimilar material system to function as a single unit, it is necessary for the loads to be transferred from one material to the next across the interfaces. It is possible that the existence of faults or cracks in one of the materials or at the interface might result in a significant increase in the local stresses, which could ultimately lead to failure if the fracture reach a critical magnitude. Because of this, it is essential to have a solid understanding of the stress state that is connected to these fissures in the dissimilar material system. The analytical stress solution exhibits a peculiar behaviour near the tip of an interface crack where the

stresses undergo a rapid reversal of sign. This highly oscillatory character takes the form $r_1^{-1/2} \sin$ (or $\cos$) of the argument $\in \log(r_1/a)$ in where a is the size of the crack, r_1 is the radial distance from the fracture border, and E is a bimaterial constant that depends on the mechanical characteristics of the materials next to it. The investigation of two-dimensional fracture issues was where this phenomenon was first uncovered [8].

The fundamental problem of two semi-infinite elastic media with different properties that are connected along a plane is considered by Erdogan [9]. Furthermore, these mediums have a series of flat holes fashioned like concentric rings. Using Green's functions for semi-infinite planes, the problem may be expressed as a system of simultaneous singular integral equations. The dominant system's closed-form solution corresponding to Cauchy kernels is shown here. The method of converting the problem to a system of linear algebraic equations is described here in order to get the whole answer, instead of solving two Fredholm integral equations. He illustrated the contact stresses with examples such as the penny-shaped fracture and bonded media with an external notch that is axially symmetric and subject to homogeneous temperature fluctuations or axial load.

Bassani and Erdogan considered the antiplane shear problem for two bonded different half planes with a semi-infinite fracture or two randomly placed collinear fractures in their work [10]. The problem is resolved for a concentrated wedge force in the semi-infinite fracture scenario, and the stress intensity factor and angular distribution of stresses are calculated. If the number of fractures is unlimited, the problem may be reduced to two integral equations. Numerical results for fractures completely embedded in homogeneous media, one crack tip touching the interface, and a crack crossing the interface are found for different fracture angles [11].

SIMULTANEOUS DUAL INTEGRAL EQUATIONS

Through the use of fractional integral operators, [12] have been able to acquire the solution to a dual integral equation that involves the G-function of two variables. For the purpose of this chapter, we will attempt to find a solution to simultaneous dual integral equations that include G-functions of two variables. According to [13-15], G-functions are defined as

$$G_{p+n, (t+v_1, t+v_2), s, (q+m_1, q+m_2)}^{n, v_1, v_2, m_1, m_2} \left| \begin{matrix} (\epsilon_n) \\ (\gamma_{t+v_1}) \\ (\gamma'_{t+v_2}) \\ (\delta_s) \\ (\beta_{q+m_1}) \\ (\beta'_{q+m_2}) \end{matrix} \right| \dots\dots\dots(1.1.7)$$

$$= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j - \xi) \prod_{j=1}^{v_1} \Gamma(\gamma_j + \xi) \prod_{j=1}^{m_2} \Gamma(\beta_j - \eta) \prod_{j=1}^{v_2} \Gamma(\gamma'_j + \eta)}{\prod_{j=1}^q \Gamma(1 - \beta_{m_1+j} + \xi) \prod_{j=1}^t \Gamma(1 - \gamma_{v_1+j} - \xi) \prod_{j=1}^q \Gamma(1 - \beta'_{m_2+j} + \eta)} \frac{\prod_{j=1}^n \Gamma(1 - \epsilon_j + \xi + \eta) \times y^n}{\prod_{j=1}^t \Gamma(1 - \gamma'_{v_2+j} - \eta) \prod_{j=1}^p \Gamma(\epsilon_{n+j} - \xi - \eta) \prod_{j=1}^s \Gamma(\delta_j + \xi + \eta)} d\xi d\eta \dots\dots\dots(1.1.8)$$

where $0 \leq m_1 \leq q$, $0 \leq m_2 \leq q$, $0 \leq v_1 \leq t$, $0 \leq v_2 \leq t$, $0 \leq n \leq p$ and $(a_r) = a_1, a_2, \dots, a_r$,
 $(a_{r+1}, s) = a_{r+1}, a_{r+2}, \dots, a_s$.

The sequence of parameters $(\beta_{m_1}), (\beta'_{m_1}), (\gamma_{v_1}), (\gamma'_{v_1})$ and (ϵ_n) do not correspond with any of the $\Gamma(\beta_j - \xi)$, $j = 1, \dots, m_1$ and $\Gamma(\beta'_j - \eta)$, $k = 1, \dots, m_2$ integer's poles. If needed, the integration pathways are indented so that all of the lie to the right and those of lie to the left of $\Gamma(\gamma_j + \xi)$, $j = 1, \dots, v_1$, $\Gamma(\gamma'_j + \eta)$, $k = 1, \dots$ and $\Gamma(1 - \epsilon_j + \xi + \eta)$, $j = 1, \dots, n$ the imaginary axis.

The integral (2. 1.1) converges if

$$\left. \begin{aligned} p+q+s+t &< 2(m_1+v_1+n) \\ p+q+s+t &< 2(m_2+v_2+n) \\ \text{and } \arg x &< \pi \left[m_1+v_1+n - \frac{1}{2}(p+q+s+t) \right] \\ \arg y &< \pi \left[m_2+v_2+n - \frac{1}{2}(p+q+s+t) \right] \end{aligned} \right\} \dots\dots\dots(1.1.9)$$

SIMULTANEOUS DUAL INTEGRAL EQUATIONS

The simultaneous dual integral equations to be discussed here are as follows:

$$\int_0^\infty \int_0^\infty G_{p+n, (t+v_1, t+v_2), 0, (q+m_1, q+m_2)}^{n, v_1, v_2, m_1, m_2} \left[\begin{array}{c} xu \\ yu' \end{array} \left| \begin{array}{c} (e_{p+n}) \\ (\gamma_{v_1}^\theta) (\gamma_{v_2}^\theta) \\ (\beta_{m_1}^\theta) (\beta_{m_2}^\theta) \\ (\gamma_{v_1+t}^\theta) (\gamma_{v_2+t}^\theta) \\ (\beta_{m_1+q}^\theta) (\beta_{m_2+q}^\theta) \end{array} \right. \right] \cdot \sum_{\psi=1}^{n'} a_{\psi\theta} f_\psi(u, u') du du' = \theta_{1\theta}(x, y), \quad 0 < x < 1, 0 < y < 1 \dots\dots(1.1.10)$$

$$\int_0^\infty \int_0^\infty G_{p+n, (t+v_1, t+v_2), 0, (q+m_1, q+m_2)}^{n, v_1, v_2, m_1, m_2} \left[\begin{array}{c} xu \\ yu' \end{array} \left| \begin{array}{c} (e_{p+n}) \\ (c) (c') \\ (b_{m_1}^\theta) (b_{m_2}^\theta) \\ (c_{t+1}^\theta) (c_{t+1}^\theta) \\ (b_{m_1+q}^\theta) (b_{m_2+q}^\theta) \end{array} \right. \right] \cdot \sum_{\psi=1}^{n'} b_{\psi\theta} f_\psi(u, u') du du' = \phi_{2\theta}(x, y), \quad x > 1, y > 1 \dots\dots(1.1.11)$$

$\theta = 1, 2, \dots, n.$

where $a_{\psi\theta}$, $b_{\psi\theta}$ are constants, $\phi_{1\theta}(x, y)$, $\phi_{2\theta}(x, y)$ are given and $f_\psi(x, y)$ is to be determined.

THEORETICAL BACKGROUND AND ANALOGY

The historical development of integral formulations began with the recognition of structural similarities between **Laplace's equation** in potential theory and the **Navier-Cauchy equations** in elasticity. Pioneers like Betti and Somigliana extended potential-theoretic techniques to elastostatics. Green's identities, particularly the third identity, play a critical role in expressing the solution inside a domain in terms of boundary data, laying the groundwork for the **boundary integral equations (BIEs)**.

INTEGRAL EQUATIONS IN POTENTIAL THEORY

In potential theory, the use of single-layer and double-layer potentials allows the transformation of Laplace and Poisson equations into boundary integrals. These formulations are well-suited to treat Dirichlet, Neumann, and Robin boundary conditions. The simplicity of the Laplace kernel ensures smooth behavior,

and the methods enjoy excellent analytical properties and convergence behavior when implemented numerically.

INTEGRAL EQUATIONS IN ELASTICITY

In elasticity, the governing equations are vector-valued and more complex due to the tensorial nature of stress and strain. Nevertheless, the use of **Somigliana's identity**, derived from Betti's reciprocal theorem, enables a similar reduction to boundary integrals. Rizzo (1967) advanced this approach by presenting a vector boundary formula connecting boundary displacements and tractions, applicable to both simply and multiply connected domains. This was a significant conceptual advancement, especially for three-dimensional problems.

NUMERICAL IMPLEMENTATION

Numerical approaches like the **Boundary Element Method (BEM)** discretize the boundary into elements, transforming the integral equations into a solvable linear system. Methods for treating singular kernels (e.g., Cauchy Principal Value) are crucial in ensuring accuracy. Modern developments also include fast multipole methods and adaptive mesh refinement to handle large-scale and complex geometries effectively.

CONCLUSION

Integral equation methods offer a powerful alternative to classical domain methods in addressing elasticity and potential theory problems. By leveraging the mathematical analogy and converting volume integrals into boundary formulations, these methods facilitate both analytical insight and numerical efficiency. With ongoing advancements in computational techniques and broader applications in engineering and physics, integral methods continue to gain prominence as a robust tool in applied mathematics and mechanics.

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