ANALYTICAL STUDY OF CERTAIN SUB CLASSES RELATED TO COMPLEX ORDER

A Thesis

Submitted towards the Requirement for the Award of the degree of

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

Under the Faculty of science

By

KUNAL SRIVASTAVA

Enrollment No: 161510606238

Under the Supervision of

Dr. DHARMENDRA BADAL

Asst Professor

P.K. UNIVERSITY

NH-27,Vill, Thanra (P.O-DINARA) Shivpuri,M.P,India-473665

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Acknowledgement

First and foremost praises and thanks to the God for his showers of abundant blessings throughout of my research work to complete the research successfully.

I am beholden to **Prof. Yogesh Dubey**, Honorable Vice Chancellor of P.K University, Shivpuri (M.P) for their cooperation and also for giving me an opportunity to do the research work at their prestigious University.

I would also like to express my sincere thank to **Dr. Jitendra Mishra**, Director Admin and also thankful to **Dr. Aiman Fatma**, Dean Academics.

My special thanks goes to **Dr. Bhasker Nalla**, Dean Research and I/c Ph.D Cell and his office Assistant **Mr. Ashish Gupta**.

I like to thank **Dr. Deepesh Namdev**, Registrar, **Ms. Nisha Yadav**, Librarian **Mr. Kamlesh Yadav** , **Mr. Alok, I T Cell** , Dr. Praveen Kumar, Faculty of Science (P.K. University, Shivpuri, M.P).

I would like to express my deep and sincere gratitude to my Research Supervisor **Dr. Dharmendra Badal** for giving me the opportunity to do research and providing invaluable guidance throughout this research. His immense knowledge, intellectual thinking, vision, dynamism and motivation have deeply inspired me. He has taught me the methodology to carry out research and to present the research work as clearly as possible. It was a great privilege and honor to work and study under the guidance

I am thankful to my family members Late **Mr. Shaswat Srivastava, Mr. Sanjat Srivastava, Mr.Sangharsh Srivastava, Dr. Puneet Mallik, and Mrs.Vandana Mallik, Mr. Lakhram Jee Mr. Sidhant Srivastava, Mr. Vedant Srivastava, Dr. Samiksha Srivastava, Mr. Anekant Srivatava, Mr. Saksham Srivastava, Mr. Prateeksha Srivastava and Mr. Srajan Srivastava, Mr. Saksham Srivastava, Mr. Devansh kartikeyan Srivastava.** I can never think accomplishing to this innovative task without affection blessings of my mother **Smt. Ramkishori Srivastava**. This work could never been **Mr.**

Mritunjay Srivastava, Principal S.J. College of Science And Arts [illegible], Mr. Dharmendra [illegible] and **Smt. Sadhana Srivastava**.

I would also like to thank my wife **Mrs. Nidhi Srivastava** and my daughter **Apoorva Srivastava**, and my son **Suyash Srivastava**. Finally, I want to thank my almighty God and my father Late **Dr. Radhe Shyam "Yogi"** without him I would unable to complete this work.

Sumit Srivastava

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ABSTRACT

The present investigation which is entitled , "**Analytical Study of Certain Sub classes related to complex order.'** will be divulging and would dig out the several conjectures and several subclasses of univalent functions using the different technique, variation methods and subordinate techniques,

Analytic function has played very important role in development geometrical theory of complex variable. Here, we describe only those aspect of theory in direction of which have purposed the study further. The present work has been made to verified and detailed study of various subclasses of analytic function by employing different techniques. In a number of cases our approach not only yields a generalization of various known results but also gives rise to may new best estimation.

We have introduced the various Certain Subclasses of Analytic function Related to Complex Order using the convolution technique belonging to the class $V(\lambda,\mu,A,B,b)$, $G(\lambda,\mu,A,B,b)$, $J(A,B,p,\delta)$. $H(A,B,p,\delta)$,$M(A.B,Z_0,\mu,\delta)$.

Tools and methodology used in research design in technique of Koebe Univalent Function, Mapping properties of analytic Function, Radius of P-valent Convexity. Preliminary lemmas are used in my thesis for solving the equations. Closure theorems are used for finding the numerical solution to the different differential equations. Univalent Function, Hadamard Product of two Analytic function, Gamma function , Biberbache conjecture, Robertson conjecture, Milin Conjecture ,Sheil- small conjecture are applied in my thesis. My research work is divided in 10 chapters.

In chapter 1, we introduced the introduction of my research work which are important in my research project and I proposed the summary of my research project We introduced a short notes of conjectures.

In chapter 2, we introduced the research problems which are necessary to our relevant study . We described the future work related my research work and find out the research problems.

In chapter 3, We introduced the $\mathbf{J^*(A,B)}$ and find out the sufficient conditions **h(z)** to be in $\mathbf{J^*(A,B)}$ and **K(A,B)** with suitable restrictions **a,b,c.** We determined the sufficient condition for **I(z)** to be in $J^*(A,B)$ with appropriate condition a,b,c. We also described

the mapping properties of $\xi(z)=zF(a,b;c;z)$ with the help of elementary results of star like Function and Convex Function.

In Chapter 4, we introduced the class **V(λ,μ,A,B,b)** using the convolution techniques .We provided some preliminary lemmas for solving the equations and displayed the contentment relation between **V(λ₀,μ,A,B,b) ⊂V(λ,μ,A,B,b)** where $\lambda_0>\lambda$. . We got maximization theorem $|a_3-\delta a_2^2|$ for the complex value of the class **V(λ,μ,A,B,b)** and we have observed the distortion properties, closure properties. We obtained the many conditions in term of coefficient for the function belonging to the class **V(λ,μ,A,B,b).**

Chapter 5, We have introduced the another class **G(λ,μ,A,B,b)** using the convolution techniques and obtained the coefficient estimate for the class G(λ,μ,A,B,b). We investigated the sufficient condition in term of coefficient. We have investigated the maximization of $|a_3-\delta a_2^2|$ for the complex value of the class **V(λ,μ,A,B,b).**

Chapter 6, We have introduced the class **J(A,B,p,δ)** in term of fractional derivative Operator $D^{\lambda} f(z)$ and obtained the sufficient condition for the claas **J(A,B,p,δ)** and Results involving the modified hadamard Product of Two Functions. We also investigated the distortion properties, we have investigated the some closure properties related my study for the class J(A,B,p,δ). We have found the radius of P-valent star likeness for the class J(A,B,p,δ). We established the contentment relation to the class **J(A,B,p,δ).**

Chapter 7, We have also introduced a new class **H(A,B,p,δ)** of analytic function defined by fractional derivative in terms of coefficient for the function *f(z)*. We have also obtained the necessary and sufficient condition, the result involving the Modified Hadamard Product of two functions, Contentment relation, p- valent convexity belonging to the class **H(A,B,p,δ)**. We have investigated Distortion properties, radius of p- valent star likeness, Closure properties to the class **H(A,B,p,δ).**

Chapter 8, We have introduced another new class M(A,B,Z_0,δ,μ) of analytic functions defined by fractional derivative having two fixed point (0,1). We have obtained the necessary and sufficient condition, Distortion Property, radius of convexity, Quesi Modified Product of two function, contentment relations to the class M(A,B,Z_0,δ,μ) We have shown the class M(A,B,Z_0,δ,μ) is closed arithmetic mean and convex linear combinations.

Chapter 9, We have investigated a new class **J(A,B,f,p,δ)** of analytic star like function, Closure property in terms of fractional Integral Operator $(D_z^{-\delta} f(z)$over the element of having negative coefficient. We have provided some Lemmas due to Goel and Sohi. We have obtained necessary and sufficient condition, contentment relation, class- preserving integral operator, Radius of convexity, Distortion Property for the class **J(A,B,f,p,δ)** . We have investigated some results involving the Modified Hadamard Product of two functions to the class **J(A,B,f,p,δ).**

Chapter 10, We have introduced the summary and conclusion through ought the research work. .

Kunal Srivastava

Enrollment No:161510606238

LIST OF SYMBOLS

1.	U	Upen Unit disc $z: \|z\| < 1$
2.	D	Domain
3.	$\in$	Belong to
4.	$k(z)$	Koebe Function
5.	Y^*	Sub class of Star like function
6.	$K(\alpha)$	The class of convex function of order α
7.	$\xi(z)$	Series representation of Class of Analytic Function$\sum_{k=2}^{\infty} a_n z^n$
8.	D^n	Differential operator
9.	Ω_z^λ	Fractional Differential Operator
10.	I(z)	Mapping properties of integral Operator
11.	h (z)	Mapping properties of analytic Function
12.	λ_n	Pochhamar Symbol
13.	$f * g$	Hadamard product (convolution) of f and g
14.	Re	Real part of complex number
15.	${}_2F_1(a,b,c;z)$	Gauss hyper geometric Function
16.	C	Complex plane
17.	S_p	Class of P-valent Function.
18.	S^*	Class of Star- like Function

TABLE OF CONTENT

Chapter-1

INTRODUCTION

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

The analytic function played a significant role in developing the geometrical theory of analytic function related to complex order. Here we described those features which are important to our further study.

In some cases, our approach is not only gave generalizations of various known outcomes but also produced many new and best estimates. The study of Certain Sub Class of Analytic Function related to Complex Order associated with defined the unit disc $U = \{z : |z| < 1\}$ and inclusion relationships to the conjugate points .

We established the some coefficients inequality for certain sub classes $G(\lambda, \mu, A, B, b)$ and some inclusion relations. The main object of analytic study of Certain Subclass of Analytic Function Related to Complex Order is to discuss the various essential properties and determine the limit of the coefficients for the classes and obtain accurate results.

In present study has been proposed to the study of a function is to be analytic in a domain D ,if it is differentiable at every point in D .Here, domain D mean is a non-empty open connected subset of the complex plane. A function f is said to be univalent in D if it is one-one in D . This takes no value more than once in D. In the other word, we can say that if $f(z_1) = f(z_2)$ at any point $z_1, z_2 \in D$ then $z_1 = z_2$ .The function $\xi(z) = z$ is analytic, but it is univalent in the complex plane. A necessary condition for an analytic to be univalent in D is that $f(z) \neq 0$,for z in D, which is not sufficient. For example, the functions ξ defined by $\xi(z) \neq e^z$ is not univalent in f through its derivative never vanishes in f .

We assume that D to be unit disc $\mu = \{z : |z| < 1\}$. By the Riemann mapping theorem , we can say that any connected domain in the complex plane is not the whole plane. It can be mapped by the analytic univalent functions on the unit disc. Thus the investigations of my analytical study of (Certain Sub Classes Related to Complex Order is univalent in a simply connected in domain D with more than one boundary point and it can be confined to the investigation of certain sub classes of analytic function which are univalent in D. This is to simplify and to give short and elegant formula, which are given below.

$$\xi(z) = z + \sum_{k=2}^{\infty} a_n z^n \tag{1.1.1}$$

. Several sub classes of Univalent Function were introduced in different techniques like parametric method, convolution techniques, variation method, subordination techniques etc, were discovered. The concept of univalence can be extended to be p-valent. A functions f is analytic in the unit disc U is said to be p-valence if the equation $w = f(z)$ has at P -solutions and there exists a w_0 in $E(z)$ has exactly P solution in U. Let P be the positive integer is greater than and equal to unity and $P(y)$ denote the class of functions of the form.

$$f(z) = z^p + \sum_{n=1}^{n} a_{p+n} z^{p+n} \tag{1.1.2}$$

The results are presented in the next seven chapters .The properties of Analytic Function defined by using hyper geometric functions. In Section 3.1, we have defined the functions $h(z)$ and $I(z)$ by (3.1.4) and (3.1.5) respectively .In section3.2,we state the lemma(3.2.1)and (3.2.2) due to , at p=1 and prove a Lemma(3.2.3) that are needed in the succeeding section .In this section 3.3,we find the sufficient conditions for $h(z)$ to be in $y^*(A.B)$.Further, we obtain the necessary and sufficient conditions for $h(z)$ to be in $y*(A,B)$ and $k(A,B)$ with appropriate restrictions on a, b and c. In section 3.4,firstly we determined the sufficient conditions for I(z) in K(A,B).Further, we also fined the necessary and sufficient condition for I(z) to be K(A,B) with appropriate restrictions on a,b,c. Our results generalize the corresponding results of Silverman .The study of the certain analytic function related to complex order ((I).The description of this chapter is divided into nine sections for the systematic of the class $V\left(\lambda,\mu,A,B,b\right)$.In the section 4.1,we have defined the class $V\left(\lambda,\mu,A,B,b\right)$.

In the section 4.2, we have stated some lemmas that are needed in the succeeding sections of this chapter. In the section 4.3, we have shown the containment relation between $V\left(\lambda_0,\mu,A,B,b\right)cV\left(\lambda,\mu,A,B,b\right)$,where $\lambda_0 > \lambda$. In section 4.4, we have obtained the coefficient estimate for the functions belonging to the class $V\left(\lambda,\mu,A,B,b\right)$.In section 4.5,we have found the sufficient condition in terms of coefficient for the function belonging to the class $V\left(\lambda,\mu,A,B,b\right)$.In section 4.6, we have obtained the maximizations of $\left|a_3-\lambda a_2^2\right|$ for complex value of δ over the class $V\left(\lambda,\mu,A,B,b\right)$.In section4.7,we have investigated the distortion properties for the class $V\left(\lambda,\mu,A,B,b\right)$.In section 4.8, we have obtained the Preserving Integral Operator of the form (4.8.1) for the class $V(\lambda,\mu,A,B,b)$.In section 4.9,we have obtained the closure property for the class $V\left(\lambda,\mu,A,B,b\right)$.

The study of another family of Certain Analytic Function Related To Complex Order(II).The description of this chapter is divided into six sections for the systematic study of the class $G\left(\lambda,\mu,A,B,b\right)$. In section 5.1,we have defined the class $G\left(\lambda,\mu,A,B,b\right)$.In section 5.2 provides some lemmas that are needed in the succeeding sections of this chapter. In sections5.3,we have obtained the coefficient estimate for the functions belonging to the class $G\left(\lambda,\mu,A,B,b\right)$.In section 5.4,we have investigated the sufficient conditions in terms of coefficient for the function belonging to the class $G\left(\lambda,\mu,A,B,b\right)$ .In section 5.5, we have determined the maximization of $\left|a_3-\delta a_2{}^3\right|$ for complex value over the $G\left(\lambda,\mu,A,B,b\right)$.In section5.6,we have found the necessary and sufficient conditions in terms of convolution for the function belonging to the class $G\left(\lambda,\mu,A,B,b\right)$.

The new family of Analytic Function defined by fractional derivative(I).The chapter (VI)consist in nine sections. In sections 6.1, we have introduced the family $J\left(A,B,p,\delta\right)$ of Analytic Function defined by fractional derivative. In section6.2, we have obtained the necessary and sufficient condition in term of coefficients for the functions is belonging to the class $J(A,B,p,\delta)$ .In section 6.3, we have obtained the distortion properties for the class $J(A,B,p,\delta)$ .In section 6.4,we have obtained the class Preserving Integral Operator of the form (6.1.5) for the class, we obtained the radius of p-valent star likeness for the class $J(A,B,p,\delta)$. In sections 6.6, we determined the radius of p-valent convexity for the class $J(A,B,p,\delta)$ .In section 6.7, we have obtained some results involving modified Hadamard Product of two Functions belonging to the class $J(A,B,p,\delta)$. In section 6.8, we obtained some contentment relations related to the $J(A,B,p,\delta)$ .In section 6.9,we have shown that the class $J(A,B,p,\delta)$ is closed under arithmetic mean and convex linear combination.

The new family of Analytic Function Defined by Fractional Derivative(II).The chapter VII consists nine sections. In sections 7.1, we have introduced the class $H\left(A,B,p,\delta\right)$.Further, we have determined the same properties for the class $H\left(A,B,p,\delta\right)$ in the same order as we have already obtained properties for the class $H\left(A,B,p,\delta\right)$ in chapter (VI). The family of Analytic Function defined by Fractional Derivative Having Two Fixed point. The chapter (VIII) consists eight section. In sections 8.1, we have defined the class $M\left(A,B,z_0,\delta,\mu\right)$. In sections 8.2, we have obtained the necessary and sufficient conditions in terms of coefficients for the functions belonging to the class $M\left(A,B,z_0,\delta,\mu\right)$.In section 8.3, we have obtained the Distortion Properties for the class $M\left(A,B,z_0,\delta,\mu\right)$.In sections 8.4, we have determined the class preserving integral operator defined by(4.8.1) for the class

$M\left(A,B,z_0,\delta,\mu\right)$.In sections 8.5, we have obtained the radius of convexity for the class $M\left(A,B,z_0,\delta,\mu\right)$. In section8.6, we have obtained some result involving Quasi-Hadamard Product of two function belonging to the class $M\left(A,B,z_0,\delta,\mu\right)$.In section 8.7, we have obtained some contentment relation related to the class $M\left(A,B,z_0,\delta,\mu\right)$.In section 8.8, we have shown the class $M\left(A,B,z_0,\delta,\mu\right)$ is closed under arithmetic mean and Convex Linear Combinations.

The family of Analytic function Defined by Fractional Integral .This chapter is divided into nine section for the systematic study of class $J\left(A,B,f,p,\delta\right)$.In section 9.1, we introduced the class $J\left(A,B,f,p,\delta\right)$.In section 9.2, we have stated the lemma due to Goel,and Sohi,needed to prove the result of succeeding sections of this chapter. In section 9.3,we have obtained the necessary and sufficient condition in term of coefficients for a function G to be in $J\left(A,B,f,p,\delta\right)$.Consequently, we have shown $J\left(A,B,f,p,\delta\right)cJ^{*}\left(A,B,p\right)$.Since $J\left(A,B,f,p,\delta\right)$ is the sub classes of $J(p)$ .It follows that the element of $J\left(A,B,f,p,\delta\right)$ are starlike and hence P-valent in U. In sections 9.4, we have obtained the contentment relation related to the class $J\left(A,B,f,p,\delta\right)$. In sections 9.5, we have obtained the class Preserving Integral Operator of the from (9.1.2) for the class $J\left(A,B,f,p,\delta\right)$.We have found the radius of p-valent star likeness of the functions G defined in (9.1.2).In section 9.6, we have found the radius of p-valent of convexity for the class $J\left(A,B,f,p,\delta\right)$.In section 9.7, we have obtained the Distortion Properties for the class $J\left(A,B,f,p,\delta\right)$.In sections 9.8,we have obtained the result involving Modified Hadamard Product of two functions belonging to the class $J\left(A,B,f,p,\delta\right)$ .In sections 9.9, we have shown that the class $J\left(A,B,f,p,\delta\right)$ is closed under Arithmetic Mean and Convex Linear Combinations..

1.2 CONVOLUTION PROPERTIES INVOLVING SUBORDINATE RELATIONS:

Now we discuss about the mapping properties for convolutions involving the HGF. In this functions in class of functions in the form.

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

Which are analytic in the open disc $U = \{z : |z| < 1\}$ and S denote the subclass of functions in A, which are univalent in U. More ever let $S^*(\alpha)$ and $K(\alpha)$ be the sub class of S consisting respectively of a functions which are starlike or order α where $0 \le \alpha < 1$ in U.

Now we describe the q-derivative operator in conjunctions with the principle of sub-ordinations between analytic functions. Recently the theory of q-analysis has attracted considerable effort from researchers. Due to its applications in many branches of mathematics and physics.

The main purpose of this theory is to introduce and study two subclasses of analytic functions in the open unit disc $U = \{z : z \in \mathbb{C} and\, |z| < 1\}$ by applying the q-derivative operator in conjunction with the principle of subordinate between analytic functions.

1.4 METHODOLOGY ADOPTED

[1]. The tools and methodology used in research design in the technique of Koebe univalent function and mapping properties of an analytic function,radius of p-valent convexity and preliminary lemmas for solving the equations.Different methods for the estimations of the operator involved in the distortion theorem, integral operator developed by Goodmann,A.W

[2]. Closure theorems are used for finding numerical solutionsto the different differential equations.

[3]. Univalent functions, Hadamard product of two analytic functions, Gamma Functions, Biberbache conjecture, Robertson conjecture,Milin Conjecture,Sheil-Small Conjecture are applied in my thesis.

CHAPTER 3

MAPPING PROPERTIES OF ANALYTIC FUNCTIONS DEFINED BY USING HYPERGEOMETRIC FUNCTION

3.1 INTRODUCTION :

Let $J^*(A,B)$ the class of those function f (z) of a A which satisfying the conditions .

$$\frac{zf'(z)}{f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, z \in U \ , \ -1 \le A < B \le 1$$

Where w(z) belonging to the class H.

-

K (A, B)= {f∈ A:$zf'(z)$∈T *(A,B)}.

We observe that

$J^*\{(2\alpha-1)\alpha,\beta\} \equiv J^*(\alpha,\beta)$, $T^*\{(2\alpha-1),1\} \equiv J^*(\alpha)$ and $J^*(-1,1)$ J^*

$K\{(2\alpha-1)\beta,\beta\} \equiv K(\alpha,\beta)$,$K\{(2\alpha-1),1\} \equiv K(\alpha)$ and K (-1,1)= $K(\alpha)$.

For a, b, c to be complex numbers with c is neither zero nor a negative integers,

$$\text{Let } F(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n (\lambda)_n} \quad . \tag{3.1.1}$$

Denote the hyper geometric function (HGF), where $(\lambda)_n$ is the Pochhammer symbol defined by

$$\lambda_n = \begin{cases} \lambda, (\lambda+1)...............(\lambda+n-1), n=1,2,................ \\ 1, n=0 \end{cases}$$

This function is analytic in unit disc U. We also that F(a,b;c;1) converge for the Re(c-a-b)>0 and it also relates to the Gamma Function which is given by

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{3.1.2}$$

Merks and Scott .Ruscheweyh and Singh studied the mapping properties of

$$\xi(z) = zF(a,b;c;z) \tag{3.1.3}$$

The method of differential subordination Recently, Silvermen investigated the mapping properties of $\xi(z)$ with the help of elementary results of starlike function and Convex Functions.

$$h(z)=(1\ \mu)\xi(z)+\mu z\xi'(z). \tag{3.1.4}$$

Where $\mu \geq 0 and \xi(z)$ defined by(3.1.3) . In fact the mapping properties of h(z)in my basic tools are lemmas due to Goal, and Sohi, at p=1 and we find out sufficient conditions for h(z) to be in J*(A,B) and $K(A,B)$.We obtain the necessary and sufficient condition for h(z) to be in J*(A,B) and $K(A,B)$ with suitable restrictions on a, b ,c.

In section (3.4), firstly, we determine the sufficient condition for $I(z)$ defined by

$$I(z)=\int_o^\infty \left\{\frac{h(t)}{t}\right\} dt \tag{3.1.5}$$

Where I(z) belong to K(A,B) .Further, we investigate the necessary and sufficient conditions $I(z)$ to be in K(A,B) with appropriate restrictions on a, b, c. Our results are generalized to the corresponding results of H. Silverman

3.2 PRELIMINARY LEMMAS :

In this section we state the lemmas [3.2.1] and [3.2.2] due to Goel,and Sohi, at p=1 and prove a Lemma [3.2.3] that are needed in our investigations

LEMMA 3.2.1: A sufficient condition for function f define by (1.1.1) to be in $J^*(A,B), K(A,B)$ is that

$$\left(\sum_{n=2}^{\infty}\{(1+B)n-(A+1)|a_n|\} \leq (B-A)\right) and \left(\sum_{n=2}^{\infty} n\{(1+B)n-(A+1)|a_n|\} \leq (B-A)\right) \tag{3.2.1}$$

LEMMA 3.2.2: A necessary and sufficient condition for f(z)

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$$

To be $J^*(A,B) and K(A,B)$, is that

$$\sum_{n=2}^{\infty}\{(1+B)n-(A+1)\}|a_n|\le(B-A) and \left(\sum_{n=2}^{\infty}n\{(1+B)n-(A+1)|a_n|\}\le(B-A)\right)$$

(3.2.2)

3.3 MAPPING PROPERTIES OF ANALYTIC FUNCTION h(z):

THEOREM 3.3.1 : If a, b>1 and c>a+b+2, then the sufficient condition for h (z) to be J* (A,B),where $1-\le A<B\le 1$ is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{\{(1+B)+\mu(1+2B-A)\}ab}{(B-A)(c-a-b-1)}+\frac{\mu(1+B)(a)_2(b)_2}{(B-A)(c-a-b-2)_2}\right]\le 2 \tag{3.3.1}$$

The condition (3.3,1) is necessary and sufficient for h_1 is defined by

$$h_1(z)=z\left[2-\frac{h(z)}{z}\right] \text{ to be in } J^*(A,B)$$

PROOF: Since

$$h(z)=z+\sum_{n=2}^{\infty}(1-\mu+\mu m)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n$$

(3.3.2)

Let according to Lemma (3.2.1) we need to show that

$$\sum_{n=2}^{\infty}\{(1+B)n-(a+1)\}(1-\mu+\mu n)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\le(B-A)$$

(3.3.3)

The left side inequality (3.3.3) converge if c>a+b+2

Now

$$=\sum_{n=2}^{\infty}\{(1+B)n-(A+1)\}(1-\mu+\mu n)\sum_{n=}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$=\sum_{n=1}^{\infty}\{(1+B)n+(B-A)\}(1+\mu n)\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$\sum_{n=2}^{\infty}\left[(1+B)\mu n^2+\{(1+B)+\mu(B-A)\}n+(B-A)\right]\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$=(1+B)\mu\sum_{n=1}^{\infty}\frac{n(a)_n(b)_n}{(c)_n(1)_{n-1}}+\{(1+B)+\mu(B-A)\}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}}+(B-A)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} \quad (3.3.4)$$

Putting $(\lambda)_n=(\lambda+1)_{n-1}$ and applying (3.1.1) and (3.1.2)

We may express that

$$\sum_{n=2}^{n}\{(1+B)-(A+1)\}(1-\mu)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}=\mu(1+B)\frac{(a)_2(b)_2}{(c)_2}F(a+2,b+2,c+2;1)$$

$$+\{(1+B)+\mu(1+2B-A)\}\frac{ab}{c}F(a+1,b+1,c+1;1)+(B-A)\{F(a,b,c;1)-1\}$$

$$=\mu(1+B)\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}+\{(1+B)+\mu(1+2B-A)\}.$$

$$.\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}-(B-A)$$

$$=(B-A)\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{\{(1+b)+\mu(1+2B)-A\}ab}{(B-A)(c-a-b-1)}+\frac{\mu(1+B)(a)_2(b)_2}{(B-A)(c)_2}\right]-(B-A)$$

Last expression is above by (B-A) if and only if holds.

Such that

$$h_1(z)=z-\sum_{n=2}^{\infty}(1-\mu-\mu n)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n \quad (3.3.5)$$

Thus this least conditions (3.3.1) is also sufficient for the h(z) to be in j*(A,B) for the Lemma (3.2.2).

COROLLARY 3.3.1: if we take $\mu=0$, then this theorem becomes, if a,b>0 and C>a+b+1,thus the sufficient condition for h_1 to be J* (A,B), $-1\le A\le B\le 1$ is that

$$\frac{\Gamma(C)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(1+B)ab}{(B-A)(c-a-b)}\right]\le 2$$

This condition is also necessary and Sufficient for h(z) defined by (3.3.5) to be in j*(A,B).

REMARKS: If we take $\mu = 0$, $A = -1$ and $B = 1$,the condition(3.3.5) is necessary and sufficient for h_1 to be in $J^*(\alpha, \beta)$

THEOREM 3.3.2 : If a, b>-1,c>0 and a, b<0 then a necessary and sufficient condition for h(z) to be in J*(A,B) is that

$$\mu(A+B)(a)_2(b)_2 + \{(1+B)\mu(1+2B-A)\}.ab(c-a-b-2) + (B-A)(c-a-b-2) \geq 0 \tag{3.3.6}$$

The condition $c \geq a+b+1-ab$ is necessary and sufficient for $h(z)$ to be in J*(A,B).

PROOF: $h(z) = z - \left|\frac{ab}{c}\right| \sum_{n=2}^{\infty} (1-\mu+\mu n) \frac{(a+1)_{n-2}(b+1)_{n-2} z^n}{(c+1)_{n-2}(1)_{n-1}}$ (3.3.7)

According to Lemma (3.2.2), we must show that

$$\sum_{n=2}^{\infty} \{(1+B)n + (A+1)\}(1-\mu+\mu n) + \frac{(a+1)_{n-2}(b+1)_{n-2} z^n}{(c+1)_{n-2}} \leq \frac{c}{ab}(B-A) \tag{3.3.8}$$

The left side of(3.3.8) converges if $c > a+b+2$

Now

$$= \sum_{n=2}^{\infty} \{(1+B)n - (A+1)\}(1-\mu+\mu n) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c-1)_{n-2}(1)_{n-1}}$$

$$= \sum_{n=0}^{\infty} \{(1+B)(n+1) + (B-A)\}\{1+\mu(n+1)\} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \left[\mu(1+B)(n+1)^2 + \{(1+B)+\mu(B-A)\}(n+1)\right] \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= \mu(1+B)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \{(1+B)+\mu(B-A)\}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$

$$+\left(B-A\right)\sum_{n=0}^{\infty}\frac{\left(a+1\right)_n\left(b+1\right)_n}{\left(c+1\right)_n\left(1\right)_{n+1}}$$

$$=\mu\left(1+B\right)\sum_{n=0}^{\infty}\frac{\left(a+1\right)_n\left(b+1\right)_n}{\left(c+1\right)_{n-1}}+\left\{\left(1+B\right)+\mu\left(1+2B-A\right)\right\}+$$

$$\sum_{n=0}^{\infty}\frac{\left(a+1\right)_n\left(b+1\right)_n}{\left(c+1\right)_n\left(1\right)_n}+\left(B-A\right)\sum_{n=0}^{\infty}\frac{\left(a+1\right)_n\left(b+1\right)_n}{\left(c+1\right)_n\left(1\right)_{n+1}}$$

$$=\mu\left(1+B\right)\frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)}\sum_{n=1}^{\infty}\frac{\left(a+1\right)_{n-1}\left(b+1\right)_{n-1}}{\left(c+1\right)_{n-1}\left(1\right)_{n-1}}+$$

$$\left\{\left(1+B\right)+\mu\left(1+2B-A\right)\right\}\sum_{n=0}^{\infty}\frac{\left(a+1\right)_n\left(b+1\right)_n}{\left(c+1\right)_n\left(1\right)_n}$$

$$+\left(B-A\right)\frac{c}{ab}\sum_{n=0}^{\infty}\frac{\left(a\right)_{n+1}\left(b\right)_{n+1}}{\left(c\right)_{n+1}\left(1\right)_{n+1}}$$

$$=\mu\left(1+B\right)\frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)}\sum_{n=0}^{\infty}\frac{\left(a+2\right)_n\left(b+2\right)_n}{\left(c+2\right)_n}+$$

$$\left\{\left(1+B\right)+\mu\left(1+2B-A\right)\right\}.\sum_{n=0}^{n}\frac{\left(a+1\right)_n\left(b+1\right)_n}{\left(c+1\right)_n}+\left(B-A\right)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{\left(a\right)_n\left(b\right)_n}{\left(c\right)_n\left(1\right)_n}$$

$$=\mu\left(1+B\right)\frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)}F\left(a+2,b+2;c+2;1\right)+$$

$$\left\{\left(1+B\right)+\mu\left(1+2B-A\right)\right\}.F\left(a+1,b+1;c+1;\right)+\left(B-A\right)\frac{c}{ab}\left[F\left(a,;b;c;1\right)-1\right]$$

$$=\mu\left(1+B\right)\frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)}\frac{\Gamma\left(c+1\right)\Gamma\left(c-a-b-2\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}+.$$

$$.\left\{\left(1+B\right)+\mu\left(1+2B-A\right)\right\}.\frac{\Gamma\left(c+1\right)\Gamma\left(c-a-b-1\right)}{\Gamma\left(c-a\right)\Gamma\left(b-a\right)}+$$

$$\left(B-A\right)\frac{c}{ab}\frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\left(c-b\right)}-\left(B-A\right)\frac{c}{ab}$$

Hence (3.3.8) is equivalent to

$$=\frac{\Gamma\left(c+1\right)\Gamma\left(c-a-b-1\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\left[\mu\left(1+B\right)\frac{\Gamma\left(a+1\right)\Gamma\left(b+1\right)}{\left(c-a-b-2\right)}\left\{1+B+\mu\left(1+2B-A\right)\right\}\right].$$

$$\frac{\left(B-A\right)}{ab}\left(c-a-b\right)\leq\left(B-A\right)\left[\frac{c}{|ab|}+\frac{c}{ab}\right]=0 \tag{3.3.9}$$

Thus (3.3.9) is valid if and only if

$$\left[\mu(1+B)\frac{(a+1)(b+1)}{(c-a-b-2)}+\{(1+B)+\mu(1+2B-A)\}+\frac{(B-A)(c-a-b-1)}{ab}\right]\leq 0$$

Which is equivalent to(3.3.6).Putting $\mu=0, A=1$ and B=1 on the condition (3.3.6),we find the condition c>a+b+1-ab which is necessary and sufficient for h(z) to be $J^{*}(A,B)$.

REMARKS: if we take $\mu=0, A=-1 and B=1$ the condition (2.3.1) is both necessary and sufficient for h_1 to be in J *(A,B).

THEOREM 3.3.3: If a, b>0 and C>a+b+3,then a sufficient conditions for h(z) to be in K(A,B), $-1\leq A<B\leq 1$ is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{\{(2+3B-A)+2\mu(1+2B-A)\}ab}{(B-A)(c-a-b)}\right]$$

$$+\frac{\{(1+B)+\mu(4+5B-A)\}}{(B-A)}\frac{(a)_2(b)_2}{(c-a-b-2)}+\frac{\mu(1+B)}{(B-A)}\frac{(a)_3(b)_3}{(c-a-b-3)_2}\Bigg]\leq 2$$

(3.3.10)

Condition (3.3.10) is necessary and sufficient for $h_1(z)$ is defined by (3.3.5) to be K(A,B).

PROOF: Since h(z) defined by (3.3.2) In view of lemma (3.2.1),we need only to show that $\sum_{n=2}^{\infty} n\{(1+B)+n-(A+1)\}(1+\mu m)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\leq(B-A)$

(3.3.11)

The left side of(3.3.11) converges if c>a+b=3

Now $\sum_{n=2}^{\infty} n\{(1+B)+n-(A+1)\}(1+\mu m)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$

$$\sum_{n=0}^{\infty}(n+2)\{(1+B)(n+2)-(A+1)\}\{1-\mu+\mu(n+2)\}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

Putting n+2=(n+1)+1, we have

$$= \sum_{n=0}^{\infty}\Big[(1+B)\mu(n+1)^3 + \big\{(1+B)\mu(1+2B-A)\big\}(n+1)^2$$

$$+\big\{(1+2B-A)+\mu(B-A)\big\}\Big]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$= \mu(1+B)\sum_{n=0}^{\infty}\left(n^2+2n+1\right)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \big\{(1+B)+\mu(1+2B-A)\big\}.$$

$$= \mu(1+B)\sum_{n=0}^{\infty}(n+1)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \big\{(1+B)+\mu(1+2B-A)\big\}\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}$$

$$.\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}\big\{(1+2B-A)\big\}+\mu(B-A).\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}+(B-A)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$= \mu(1+B)\sum_{n=0}^{\infty}\frac{n(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \big\{(1+B)+\mu(3+4B-A)\big\}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$+\big\{(2+3B-A)+2\mu(1+2B-A)\big\}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + (B-A)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \mu(1+B)\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} + \big\{(1+B)+\mu(3+4B-A)\big\}.$$

$$\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} + \big\{(2+3B-A)+\mu(3+4B-A)\big\}.\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}(B-A)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \mu(1+B)\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+1}(1)_{n+1}} + \big\{(1+B)+\mu(4+5B-A)\big\}\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+1}(1)_n} +$$

$$+\big\{(2+3B-A)+2\mu(1+2B-A)\big\}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \big\{(B-A)\big\}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \mu(1+B)\sum_{n=0}^{\infty}\frac{(a)_{n+3}(b)_{n+3}}{(c)_{n+3}(1)_n} + \big\{(1+B)+\mu(4+5B-A)\big\}.\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} +$$

$$\big\{(2+3B-A)+2\mu(1+2B-1)\big\} + \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + (B-A)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

Since $(a)_{n+k} = (a)_k(a+k)_n$.

We may write the (3.3.12) as

$$= \mu(1+B)\frac{(a)_3(b)_3}{(c)_3}\sum_{n=0}^{\infty}\frac{(a+3)_n(b+3)_n}{(c+3)_n(1)_n} + \big\{(1+B)+\mu(4+5B-A)\big\}.$$

$$\cdot\frac{(a)_2(b)_2}{(c)_2}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}\left\{(2+3B-A)+2\mu(1+2B-A)\right\}.$$

$$\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(B-A)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-(B-A)$$

$$=\mu(1+B)\frac{(a)_3(b)_3}{(c)_3}F(a+3,b+3;c+3;1)+\left\{(1+B)+\mu(4+5B-A)\right\}\ .$$

$$\cdot\frac{(a)_2(b)_2}{(c)_2}F(a+2,b+2,c+2,1)+\left\{(2+3B-A)+2\mu(1+2B-A)\right\}.$$

$$\frac{ab}{c}F\left(a+1,b+1;c+1;1\right)(B-A)F(a,b;c;z)-(B-A)$$

$$=\mu(1+B)\frac{(a)_3(b)_3}{(c)_3}\frac{\Gamma(c+3)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\left\{(1+B)+\mu(4+5B-A)\right\}.$$

$$\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}+\left\{(2+3B-A)+2\mu(1+2B-A)\right\}.$$

$$\frac{ab}{c}\frac{\Gamma(c-a)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+(B-A)\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-(B-A)$$

$$=(B-A)\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)}\left[1+\frac{\left\{(1+2B-A)+2\mu(1+2B-A)\right\}ab}{(B-A)(c-a-b-1)}\right.$$
$$\left.+\frac{\left\{(1+B)+\mu(4+5B-A)\right\}}{(B-A)}\frac{(a)_2(b)_2}{(c-a-b-2)_2}+\frac{\mu(1+B)(a)_3(b)_3}{(B-A)(c-a-b-3)}\right]-(B-A)$$

This last expression is bounded above by $(B-A)$ if and only if (3.3.10) holds. Since $h_1(z)$ is defined by (3.3.5).The condition (3.3.10) is also necessary for $h_1(z)$ to be in K (A, B) from Lemma (3.2.2).

COROLLARY 3.3.4: If we take $\mu=0$, then Theorem (3.3.2) becomes. If we take it $a,b>-1, c>0$ and $ab<0$, then a necessary and sufficient condition for h(z) to be in $J^*(A,B)$ is that $c>a+b+1-(A+B)ab/(B-A)$. The condition $c>a+b+1-ab$ is necessary and sufficient for h(z).

THEOREM 3.3.4 : If a, b>-1 a b>0 and c>a+b+3, then a necessary and sufficient condition for h (z) to be in K(A,B)is that

$$\mu(1+B)(a)_3(b)_3+\{(1+B)+\mu(4+5B-A)\}(a)_2(b)_2(c-a-b-3)+$$

$$+\{(2+3B-A)+2\mu(1+2B-A)\}ab(c-a-b-3)_2+(B-A)(c-a-b-3)_3\geq 0$$

PROOF: Since h(z) is defined by (3.3.7) .In view of Lemma (3.2.2) ,

We must show that

$$\sum_{n=2}^{\infty} n\{(1+B)n-(A+1)\}(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+2)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\leq\left|\frac{c}{ab}\right|(B-A)$$

The left side of (3.3.14) converges if $c>(a+b+3)$

Now

$$\sum_{n=2}^{\infty} n\{(1+B)n-(A+1)\}(1-\mu+\mu n)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$=\sum_{n=0}^{\infty}(n+2)\{(1+B)(n+2)-(A+1)\}\{1-\mu+\mu(n+2)\}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$

Writing n+2=(n+1) +1 , we have

$$=\sum_{n=0}^{\infty}\Big[\mu(1+B)(n+1)^3+\{(1+B)+\mu(1+2B-A)\}(n+1)^2$$

$$+\{(1+2B-A)+\mu(B-A)\}(n+1)+(B-A)\Big]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\mu(1+B)\sum_{n=0}^{\infty}(n^2+2n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$

$$+\{(1+B)+\mu(1+2B-A)\}\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$

$$+\{(1+2B-A)+\mu(B-A)\}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(B-A)\sum_{n=0}^{\infty}\frac{(a+1)_n(a+1)_n}{(a+1)_n(1)_n}$$

$$=(1+B)\sum_{n=0}^{\infty} n\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n-1}}+\{(1+B)+\mu(3+4B-A)\}.$$

$$.\sum_{n=0}^{\infty} n\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n-1}}+\{(2+3B-A)+2\mu(1+2B-A)\}.$$

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(B-A)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$.=\mu(1+B)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_{n+1}(b+1)_{n+1}}{(c+1)_{n+1}(1)_n}+\{(1+B)+\mu(3+4B-A)\}.$$

$$.\sum_{n=0}^{\infty}\frac{(a+1)_{n+1}(b+1)_{n+1}}{(c+1)_{n+1}(1)_n}+\{(2+3B-A)+2\mu(1+2B-A)\}.$$

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(B-A)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_{n+1}}$$

$$=\mu(1+B)\sum_{n=o}^{\infty}\frac{(a+1)_{n+1}(b+1)_{n+1}}{(c+1)_n(1)_{n-1}}+\{(1+B)+\mu(4+5B-A)\}.$$

$$.\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_{n+1}(1)_n}+\{(2+3B-A)+2\mu(1+2B-A)\}.$$

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_{n+1}(1)_n}+(B-A)\frac{c}{ab}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$=\mu(1+B)\sum_{n=0}^{\infty}\frac{(a+1)_{n+2}(b+1)_{n+2}}{(c+1)_{n+2}(1)_n}+\{(1+B)+\mu(4+5B-A)\}.$$

$$.\sum_{n=0}^{\infty}\frac{(a+1)_{n+1}(b+1)_{n+1}}{(c+1)_{n+1}(1)_n}+\{(2+3B-A)+2\mu(1+2B-A)\}.$$

$$.\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n}+(B-A)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$=\mu(1+B)\frac{(a+1)_2(b+1)_2}{(c+1)_2}\sum_{n=0}^{\infty}\frac{(a+3)_n(b+1)_n}{(c+1)_n(1)_n}+\{(1+B)+\mu(4+5B-A)\}.$$

$$.\frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_2(b+2)_n}{(c+3)_n(1)_n}+\{(2+3B-A)+2\mu(1+2B-A)\}.$$

$$.\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(B-A)\frac{c}{ab}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-(B-A)\frac{c}{ab}$$

$$.\{(2+3B-A)+2\mu(1+2B-A)\}F(a+1,b+1;c+1;1)+$$

$$(B-A)\frac{c}{ab}F(a,b;c;1)-(B-A)\frac{c}{ab}$$

$$=\mu(1+B)\frac{(a+1)_2(b+1)_2}{(c+1)_2}\frac{\Gamma(c-3)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}$$

$$+\{(1+B)+\mu(4+5B-A)\}\frac{(a+1)(b+1)}{(c+1)}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}$$

$$+\{(2+3B-A)+2\mu(1+2B-A)\}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}$$

$$+(B-A)\frac{c}{ab}\frac{\Gamma(b-a)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-(B-A)\left(\frac{c}{ab}\right)$$

$$=\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[\mu(A+B)\frac{(a+1)_2(b+1)_2}{(c-a-b-3)}+\{(1+B)+\mu(4+5b-A)\}.$$
$$(a+1)(b+1)+\{(2+3B-A)+2\mu(1+2B-A)\}(c-a-b-2)$$
$$+(B-A)\frac{(c-a-b-2)_2}{ab}\Bigg]-(B-A)\frac{c}{ab}.$$

This last expression is bounded above by $\left|\frac{c}{ab}\right|(B-A)$ if and only if

$$=\Bigg[\mu(1+B)\frac{(a+1)_2(b+1)_2}{(c-a-b-3)}+\{(1+B)+\mu(4+5B+A)\}(a+1)(b+1)$$
$$+\{(2+3B-A)+2\mu(1+2B-A)\}(c-a-b-2)+(B-A)+\frac{(c-a-b-2)_2}{ab}\Bigg]\le 0$$

Which is equivalent to (3.3.13).

COROLLARY 3.3.5: If we take $\mu=0, A=(2\alpha-1)\beta$ and $B=\beta$,then the theorem (3.3.2) becomes. If $a,b>-1, c>0$ and $ab<0$, then a necessary and sufficient conditions for h(z) to be in $J^*(\alpha,\beta)$ is that

$$c\ge a+b+1-(1+\beta)ab/2\beta(1-\alpha)\ .$$

The condition $c>a+b+1-ab$ is necessary and sufficient for h(z) to be in J^* .The following theorem are parallel to the theorems (3.3.1) and(3.3.2) for the convex case.

COROLLARY 3.3.6: If we take μ=0,then the theorem (3.3.3) becomes: If a, b>0 and c>a+b+2, then a sufficient conditions for $h_1(z)$ to be in K(A,B) $-1\le A\le B\le 1$ is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(2+3B-A)}{(B-A)}\frac{ab}{(c-a-b-1)}+\frac{(1+B)(a)_2(b)_2}{(B-A)(c-a-b-2)_2}\right]\le 2$$

This condition is necessary and sufficient for $h_1(z)$ defined by(3.3.5) to be in corollary in K(A,B) .

COROLLARY 3.3.7: If we take **.**If we take μ=0 , $A=(2\alpha-1)\beta$ and B=β, then Theorem (3.3.3) becomes:

If a, b >0 and c>a+b+2, then a sufficient conditions $h_1(z)$ to be in K(α, β),0≤α<1,0<β≤1 is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(2+4\beta-2\alpha\beta)}{2\beta(1-\alpha)}\frac{ab}{(c-a-b-1)}+\frac{(1+\beta)(a)_2(b)_2}{2\beta(1-\alpha)(c-a-b-2)_2}\right]\leq 2$$

This condition is necessary and sufficient for $h_1(z)$ is defined by(3.3.5) to be in K(A,B).

COROLLARY 3.3.7: If we take μ=0,then the theorem (3.3.4) becomes: If a, b>-1,ab<0 and c>1+b+2, then a necessary and sufficient condition for h(z) to be in K(A,B) is that

$$\left[(1+B)(a)_2(b)_2+(2+3B-A)ab(c-a-b-2)+(B-A)(c-a-b-2)_2\right]\geq 0$$

COROLLARY 3.3.8: If we take $\mu=0, A=(2\alpha-1)\beta$ and $B=\beta$ then theorem (3.3.4) becomes: If a, b>0,ab<0 and c>a+b+2, then a necessary and sufficient condition for h(z) to be in K(α,β) is that

$$\left[(1+\beta)(a)_2(b)_2+(2+4\beta-2\alpha\beta)ab(c-a-b-2)+2\beta(1-\alpha)(c-a-b-2)\right]\geq 0$$

3.4 MAPPING PROPERTIES OF INTEGRAL OPERATOR I(z):

THEOREM 3.4.1: If a, b>0 and c>a+b+2, then a sufficient condition for I(z) defined by (3.1.6) to be in K(A,B),-1<A<B<1, is that

$$\frac{\Gamma(2)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{\{(1+B)+\mu(1+2B-A)\}ab}{(B-A)(c-a-b-1)}+\frac{\mu(1+B)(a)_2(b)_2}{(B-A)(c-a-b-2)_2}\right]\leq 2 \quad (3.4.1)$$

PROOF: With the help of Lemma (3.2.3) and theorem (3.3.1),the proof is obvious.

COROLLARY 3.3.4: If we take μ=0,then Theorem (3.1.1) becomes: If a, b>0 and c>a+b+2 , then a sufficient condition for I(z) defined by (3.1.6) to be inK(A,B),-1<A<B<1isthat $\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(1+B)ab}{(B-A)(c-a-b-1)}\right]\leq 2$

(3.4.2)

COROLLARY 3.3.2: If we take μ=0 ,then the theorem (3.1.1) is a sufficient for I(z) defined by (3.1.6) to be in K(α,β), 0≤b<1,0<β≤1 is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(1+\beta)ab}{2\beta(1-\alpha)(c-a-b-1)}\right]\leq 2$$

THEOREM 3.4.2: If a, b>-1,a,b<0 and c>a+b+2 , then a necessary and sufficient conditions for I(z) to be in K(A,B) is that

$$\left[\mu(1+B)(a)_2(b)_2+\{(1+B)+\mu(1+2B-A)\}ab(c-a-b-2)+(B-A)(c-a-b-2)_2\right]\geq 0$$

COROLLARY 3.4.3: If we take μ=0, then the theorem(3.4.2) becomes. If a, b>-1,ab<0 and c>a+b+1, then a necessary and sufficient conditions for I(z) to be in K(A,B) is that

$$c\geq a+b+1-\frac{(1+B)ab}{(B-A)}$$

COROLLARY 3.4.4: If we take μ=0 ,A=(2α-1)β and B=β then the theorem (3.4.2) becomes: If a, b>-1 ab<0 and c>a+b+2 , then a necessary and sufficient conditions for I(z) to be in K(α, β) is that c≥a+b+1-ab(1+β)/2β(1-α)

REMARK: If we take μ=0 ,A=(2α-1) and B=1 , our results concede with corresponding results of Silverman

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CHAPTER 4

CERTAIN SUB CLASSES OF ANALYTIC FUNCTIONS RELATED TO COMPLEX ORDER (I)

4.1 INTRODUCTION:

If f and g are any two functions in such that f and g defined by (1.1.1) by

$$f(g) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{And} \quad g(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

Then the Convolution Techniques or Hadamard products of f and g is denoted by $f * g$ is defined by the power series.

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

Now, we have introduced the class $V(\lambda, \mu, A, B, b,)$ of the Analytic Functions of Complex Order b, by using the Convolutions Technique, as defined below. A function f of A belongs to the class $V(\lambda, \mu, A, B, b)$.If and only if there exists a functions W belonging to the class x such that.

$$1 + \frac{1}{b}\left(\frac{D^{\lambda+1} f(z)}{z} - 1\right) = (1-\mu) + \mu\left(\frac{1+Aw(z)}{1+Bw(z)}\right), z \in U, \qquad (4.1.1)$$

Where $-1 \le B < A \le 1, 0 < \mu \le i, \lambda > -1$

$$D^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+l}} * f(z) = \frac{z\left(z^{\lambda-1} f(z)\right)^{\lambda}}{\lambda!}$$

Where . Ruchewyeh] observed that

$$D^{\lambda} f(z) = \frac{z\left(z^{\lambda-1} f(z)\right)^{\lambda}}{\lambda!}$$

It is easy to see that the conditions (4.1.1) is equivalent to

$$\left| \frac{\dfrac{D^{\lambda+1} f(z)}{z} - 1}{\mu(A-B)b - \left\{B \dfrac{D^{\lambda+1} f(z)}{z} - 1\right\}} \right| < 1, z \in \mu$$

By giving the specific values to A, B and b in (4.1.3). We obtain the following sub classes studied by the various researchers in earlier works.

I. For μ=1, A=1 and B=-1, we obtain the class of the functions f is satisfying the condition.

$$\left|\frac{\frac{D^{\lambda+1}f(z)}{z}-1}{2b-B\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}\right|<1, z\in\mu$$

For $b=(\cos\alpha)e^{i\alpha}$,we obtain the class of function f satisfying the Condition.

$$\left|\frac{e^{i\alpha}\frac{D^{\lambda+1}f(z)}{z}-1}{\mu(A-B)b-Be^{i\alpha}\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}\right|<1, z\in\mu$$

Where $\alpha\in\left(-\frac{n}{2},\frac{n}{2}\right)$ For μ=1 and b=1, we obtain the class of function f is satisfying the condition.

$$\left|\frac{\frac{D^{\lambda+1}f(z)}{z}-1}{A-B\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}\right|<1, z\in\mu$$

For μ=1,A=1-2α,B=-1 and b=1,We obtain the class of functions f satisfying the condition.

$$\left|\frac{\frac{D^{\lambda+1}f(z)}{z}-1}{\frac{D^{\lambda+1}f(z)}{z}-(1+2\alpha)}\right|<1, z\in\mu$$

Where $0\le\alpha<1$,

II. For μ=1 and $\lambda=0$.We obtain the class of functions f satisfying the conditions.

$$\left|\frac{f^{'}(z)-1}{b(A-B)-B\{f'(z)-1\}}\right|<1, z\in\mu$$

Where $D\left(f^{'}(z)\right)=f^{'}(z)$.This function is studied by Dixit ,K.K and Pal, S.K [58].

III. For μ=1 and b= (cosα) $e^{i\alpha}$ and $\lambda=0$, we obtain the class of function f is satisfying the conditions.

$$\left|\frac{e^{i\alpha}f'(z)-1}{Be^{i\alpha}f'(z)-A\cos\alpha+iB\sin\alpha}\right|<1, z\in\mu$$

Where $\alpha\in\left|-\frac{n}{2},\frac{n}{2}\right|$ is studied by Shukla, and Dashrath For μ=1,b=1, $\lambda=0$,we obtain the class of function f is satisfying the conditions.

$$\left|-\frac{f'(z)-1}{\{Bf(z)-A\}}\right|<1, z\in\mu$$

This is studied by Goel, and Mahrok For μ=1,A=δ ,B=δ ,b=1 and $\lambda=0$,we get the class of function f is satisfying the conditions:

$$\left|\frac{f'(z)-1}{f'(z)-1}\right|<\delta, z\in\mu$$

, For μ=1,A=(1-2P)δ ,B=-δ b=1 and $\lambda=0$ we get the satisfying the condition:

$$\left|\frac{f'(z)-1}{f(z)+1-2p}\right|<\delta, z\in\mu$$

Where $0\le P<,0<\delta\le 1$ this way, we come to near the study of the class $V(\lambda,\mu,A,B,b)$. The study of my thesis possesses nine parts, In part 4.2, we provide some lemmas that have necessity in the succeeding sections of this chapter. In the part 4.3,we have displayed the containment relation between $V(\lambda_0,\mu,A,B,b)\,\mathrm{c}\,V(\lambda,\mu,A,B,b)$ where $\lambda_0>\lambda$. In the section 4.4,we have received the coefficient for the function f(z) belonging to the class $V(\lambda,\mu,A,B,b)$.In section 4.5,we have obtained the many condition in term of coefficient for the function f belonging to the class $V(\lambda,\mu,A,B,b)$.

In the part4.6,we got the maximization of $\left|a_3 - \delta a_2\right|$ for the complex value of over the class $V(\lambda, \mu, A, B, b)$.In section4.7,we have observed distortion properties of class $V(\lambda, \mu, A, B, b)$.In part 4.8,we have investigate the class Preserving Integral Operator of the form(4.8.1) for the class $V(\lambda, \mu, A, B, b)$.In the section 4.9,we have received the Closure Properties for the class $U(\lambda, \mu, A, B, b)$.

4.2 PRELIMINARY LEMMAS:

In this part, we describe the Lemma 4.2.1 and 4.2.2 and prove that the lemma (4.2.3) that our need in our observation.

LEMMA4.2.1: If a function W is analytic for $|z| \le r < 1, w(0) = 0$ and

$$\left|w(z)_0\right| = \max |z| = r\left|w(z)\right|,$$

Then

$$z_0 w'(z_0) = w(z_0) \tag{4.2.1}$$

LEMMA 4.2.2: Let $W(z) = \sum_{k=1}^{\infty} c_k z^k$ be analytic with $|w(z)| < 1$ in U. If d is any complex number, then

$$\left|c_2 - dc_2^{\,2}\right| \le \max\left\{1, |d|\right\} \tag{4.2.2}$$

Equality may be attained with function $W(z) = z^2$ and $W(z) = z$

LEMMA 4.2.3: A function f belong to the class $V(\lambda, \mu, A, B, b)$, $-1 < B < A \le 1$. If and only if

$$\left|H(z) - m\right| < M, z \in u, \tag{4.2.3}$$

Where

$$H(z) = 1 + \frac{1}{b}\left|\frac{D^{\lambda+1} f(z)}{z} - 1\right| \tag{4.2.4}$$

$$m = 1, -\frac{B\mu(A-B)}{\left(1-B^2\right)}$$

and

$$m=\frac{\mu\left(A-B\right)}{\left(1-B^{2}\right)}$$

(4.2.5)

PROOF: Suppose that $f\in V\left(\lambda,\mu,A,B,b\right)$. Then form (4.1.1) .We get

$$H(z)=\frac{1+B+\mu\left(A-B\right)W(z)}{1+BW(z)}$$

There fore

$$H(z)-m=\frac{1-m+B+\mu\left(A-B\right)-BmW(z)}{1+BW(z)} \qquad (4.2.6)$$

$$=M\,\frac{B+w(z)}{1+BW(z)}=Mh(z)$$

It is clear that the function h (z) satisfies $\left|n\left(z\right)\right|<1$ .Hence (4.2.3) follows from(4.2.7).

Conversely, suppose that the condition (4.2.7) holds.

Then we have

$$\left|\frac{H\left(z\right)}{M}-\frac{m}{M}\right|<1$$

$$\frac{\rho\left(z\right)-\rho\left(0\right)}{1-\rho\left(0\right)\rho\left(z\right)}$$

Clearly W (0) =0 and $\left|W\left(z\right)\right|<1$

Note- The condition (4.2.3) can be written as

$$\left|\frac{\left(1-B\right)H\left(z\right)-1+\mu\left(A-B\right)}{\mu\left(A-B\right)}-\frac{1}{1+B}\right|<\frac{1}{1+B},z\in u$$

Now as B<<<<-1, the above condition reduced to

$$\operatorname{Re}\left\{H(z)\right\}>\frac{1}{2}\left\{2-\mu\left(1+A\right)\right\},z\in u,$$

Which is covalent to (4.2.1) when B=-1. Thus including the limiting case B<<<-1 The results proved with the help of above Lemma will hold for $-1\le B<A\le 1$. Throughout this chapter H (z), m, M are given by (4.2.4), (4.2.5) and (4.2.6) respectively.

4.3 CONTENTMENT RELATION:

THEOREM 4.3.1 Let λ_0 be any integer such that $\lambda_0 > \lambda$.

Then

$$V(\lambda_0, \mu, A, B, b) c V(\lambda, \mu, A, B, b)$$

(4.3.1)

PROOF: In order to established the required result .It is show that

$$V(\lambda+1, \mu, A, B, b) c V(\lambda, \mu, A, , b).$$

Let $f \in V(\lambda+1, \mu, A, B, b)$, choose the function such that

$$H(z) = \frac{1+\{B+\mu(AB)\}W(z)}{1+BW(z)} \tag{4.3.2}$$

Where W (0) =0 and w (z) is either analytic or meromorphic in u. It is easy to verify that

$$f(D^{\lambda+1} f(z)) = (\lambda+2) D^{\lambda+2} f(z) - (h+1) D^{\lambda+1} f(z) \tag{4.3.3}$$

Differentiating (4.3.2) and using (4.3.3)we get

$$1+\frac{1}{b}\left(\frac{D^{\lambda}+2f(z)}{z}-1\right)(-m) \tag{4.3.4}$$

$$= \frac{(1-m)+\{B+u(A-B)-3m\}w(z)}{1+Bw(z)} + \frac{u(A-B)}{\lambda+2}\frac{zw'(z)}{(1+Bw(z))}$$

Let r^* be the distance from the origin to the pole of W(z) is nearest to the origen. Thus W (z) is analytic in the disc$|z| < r_o = \min(r^*, 1)$ .By the Lemma4.2.1, for$|z| \le r(r \le r_0)$, there exist the point Z_0 such that

$$Z_o W'(Z_0) = W(Z_0) \ge 1$$

(4.3.5)

From (4.3.4) and (4.3.5) ,we have

$$\left[1+\frac{1}{b}\left\{\frac{D^{\lambda+2} f(z_0)}{z_0}-1\right\}\right] - m = \frac{N(z_0)}{R(z_0)} \tag{4.3.6}$$

Where

$$N(z_0) = (1-m)(\lambda+2) + \left[(1-M)(\lambda+2)B + \{B+u(A-B)-Bm\}\right]$$

$$(\lambda+2) + u(A-B)]W(Z_o) + B(\lambda+2)\{B+u(A-B)-Bm\}W^2(Z_0)$$

And

$$R(Z_0)=(\lambda+2)\{1+2BW(Z_0)+B^2W^0(Z_0)\}.$$

Now we suppose that it was possible to have

$$M(r,W)=\max|W(z)|=1|z|=r$$

For some $r<r_0<1$. At the point Z_0, where this occurs, we would have $|W(Z_0)|=1$. Then using identity $1.m=BMadbB+\mu(A-B)-Bm=M$, we have.

$$\left|N(Z_0)^2\right|^2-M^2|rz_0|^2=a+2\beta\operatorname{Re}\{W(z_0)\}$$

Where

$$a=\mu(A-B)\in\left[\mu(A-B)+2(\lambda+2)M(1+B)^2\right]$$

and

$$\beta=2\varepsilon\mu(A-B)MB(\lambda+2) \tag{4.3.7}$$

$$|N(Z_0)|^0-M^2|R(Z_0)|^2>0$$

Provided a±2β>0

Now, In view of the fact $\mu(A-B)>0$, It n follows that

$$\alpha+2\beta=\mu(A-B)\varepsilon\left|\mu(A-B)\varepsilon+2(\lambda+2)M(1+B)^2\right|>0$$

and

$$\alpha+2\beta=\mu(A-B)\varepsilon\left|\mu(A-B)\varepsilon+2(\lambda+2)M(1+B)^2\right|<0$$

Thus from (4.3.6) and (4.3.8) .We get But this is contrary to (4.2.4).So, we cannot have $M(r,w)=1$. Thus $|W(z)|=1$ in $|z|<0$. Since W(0)=0, $|W(z)|$. It cannot have a pole at $|z|=r_0$. Therefore W is analytic in μ and satisfies belong to $V(\lambda,\mu,A,B,b)$.

4.4 COEFFICIENT ESTIMATE:

THEOREM 4.4.1 : If function f defined by (1.1.1) belong to the class $V(\lambda,\mu,A,B,b)$ then

$$|a_n| \le \left|\frac{\mu(A-B)|b|}{\alpha(\lambda,n)}\right|, n=2,3... \qquad (4.4.1)$$

Where

$$\alpha(\lambda,n)=\binom{\lambda+n}{\lambda+1}. \qquad (4.4.2)$$

The inequality (3.4.1) is sharp.

PROOF: Since $f \in V(\lambda,\mu,A,B,b)$. We have form (4.1.1)

$$\frac{D^{\lambda+1}f(z)}{z}=\frac{1+\{\mu(A-B)b+B\}w(z)}{1+Bw(z)} \qquad (4.4.3)$$

Where w belonging to the class H .

From (4.4.3), we have

$$\frac{D^{\lambda+1}f(z)}{z}-1+\left[\mu(A-B)b+B\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}\right]w(z)$$

or

$$\sum_{j=2}^{\infty}\alpha(\lambda,j)a_j z^{j-1}=\left[\mu(A-B)b+B\sum_{j=2}^{\infty}\alpha(\lambda,j)a_j z^{j-1}\right] \qquad (4.4.4)$$

Where

$$W(z)=\sum_{j=1}^{\infty}t_j z^j$$

Equating the corresponding coefficients on the both sides of (4.4.4), we find that the coefficients on the left hand side of the (4.4.4) depends only $a_2, a_3.....a_{n-1}$ on the right hand side of(4.4.4). Hence for $h \ge 2$. It follows from (4.4.4) that

$$\sum_{j=2}^{\infty}\alpha(\lambda,j)a_j z^{j-1}+\sum_{j=n+1}^{\infty}d_j^{j-1}\left[\mu(A-B)b+B\sum_{j=2}^{n-1}\alpha(\lambda,j)a_j z^{j-1}\right]w(z)$$

Here d_j are some complex numbers. Since $|W(z)| < 1,$ by using Perceval Identity, we get

$$\sum_{j=2}^{\infty}\{\alpha(\lambda, j)\}^2 |a_j|^2 r^{2(j-1)} + \sum_{n+1}^{\infty}|d_j|^2 r^{2(j-1)}$$

$$\leq \mu^2 (A-B)^2 |b|^2 + B^2 \sum_{j=2}^{n-1}\{\alpha(\lambda, j)\}^2 |a_j|^2 r^{2(j-1)}$$

$$\leq \mu^2 (A-B)^2 |b|^2 + B^2 \sum_{j=2}^{n-1}\{\alpha(\lambda, j)\}^2 |a_j|^2$$

Taking(r-1) on the left hand side of the above inequality, we have obtain the result

$$\sum_{j=2}^{\infty}\{\alpha(\lambda, j)\}^2 |a_j|^2 \leq \mu^2 (A-B)^2 |b|^2 + B^2 \sum_{j=2}^{n-1}\{\alpha(\lambda, j)\}^2 |a_j|^2$$

Thus

$$\{\alpha(\lambda, n)\}^2 [a_n]^2 \leq \mu^2 (A-B)^2 |(b)|^2 - (1-B^2) n - \sum_{j=2}^{n-1}\{(\lambda, j)\}^2 |a_j|^2 \leq \mu^2 (A-B)^2 |b|^2$$

Hence

$$|a_n| \leq \frac{\mu(A-B)|d|}{\alpha(\lambda, n)} n = 2,3...$$

In order to established the sharpness, we consider the function f is given by

$$1+\frac{1}{b}\left(\frac{D^{\lambda} f(z)}{z}-1\right) = (1-\mu) + \mu\left(\frac{1+Az^{n-1}}{1+Bz^{n-1}}\right), n = 2,3...$$

We observe that

$$\left|\frac{\left\{\dfrac{D^{\lambda+2} f(z)}{z}-1\right\}}{\mu(A-B)b - B\left\{\dfrac{D^{\lambda+2} f(z)}{z}-1\right\}}\right| < 1$$

Hence $f \in (\lambda+1, \mu, A, B, b)$. It is easy to compute that the function f has the expansion.

$$f(z) = z + \frac{\mu(A-B)|b|}{\alpha(\lambda, n)} z^n + ...$$

Showing that the estimate (4.4.1) is sharp.

THEOREM 4.4.2: If f is defined by (1.1.1) belongs to the class $V(\lambda+1,\mu,A,B,b)$,then

$$\left(1-B^2\right)\sum_{j=2}^{\infty}\left\{\alpha\left(\lambda,j\right)\right\}^2\left|a_j\right|^2 \le \mu^2\left(A-B\right)^2\left|b\right|^2 \qquad (4.4.5)$$

Where $\alpha\left(\lambda,j\right)$ is according to (4.4.2).

PROOF: Since $f\in V\left(\lambda+1,\mu,A,B,b\right)$,we have

$$1+\frac{1}{b}\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}=(1-\mu)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}$$

Where $w(z)=\sum_{j=1}^{\infty}t_j z^j$ is analytic in u and satisfies $w\left(0\right)=0$ and $\left|w\left(z\right)\right|<1$

for $z\in u$ .Hence

$$\sum_{j=2}^{\infty}\alpha\left(\lambda,j\right)a_j z^{j-1}=\left[\mu\left(A-B\right)b-B\sum_{j=2}^{\infty}\alpha\left(\lambda,j\right)a_j z^{j-1}\right]w\left(z\right)$$

On solving, we get

$$\sum_{j=2}^{\infty}\left\{\alpha\left(\lambda,j\right)\right\}^2\left|a_j\right|^2 \le \mu^2\left(A-B\right)^2\left|b\right|^2+B\sum_{j=2}^{n-1}\left\{\alpha\left(\lambda,j\right)\right\}^2\left|a_j\right|$$

Or

$$\left(1-B\right)^2\sum_{j=2}^{\infty}\left\{\alpha\left(\lambda,j\right)\right\}^2\left|a_j\right|^2 \le \mu^2\left(A-B\right)^2\left|b\right|^2$$

4.5 SUFFICIENT CONDITION:

THEOREM 4.5.1: Let the function f is defined by(1.1.1) be analytic in u .If, for $1-\le B<0$, $\sum_{n=2}^{\infty}\left(1-B\right)\alpha\left(\lambda,n\right)\left|a_n\right|\le\mu\left(A-B\right)\left|b\right|$,where $\alpha\left(\lambda,n\right)$ is defined by(4.4.2),then f belongs to the class $V\left(\lambda+1,\mu,A,B,b\right)$.The result is sharp. Although converse need not to be true.

PROOF: Suppose that the inequality (4.5.1) holds. Then, f or $z\in u$,

We have

$$\left|\frac{D^{\lambda+1}f(z)}{z}\right|-\left|\mu(A-B)b+B\left\{\frac{D^{\lambda+1}f(z)}{z}\right\}\right|$$

$$\left|\sum_{n=2}^{\infty}\alpha(\lambda,n)a_n z^{n-1}\right| - \left|\mu(A-B)b + B\sum_{n=2}^{\infty}\alpha(\lambda,n)a_n z^{n-1}\right|$$

$$\leq \sum_{n=2}^{\infty}\alpha(\lambda,n)|a_n|r^{n-1} - \left\{\mu(A-B)|b| + B\sum_{n=2}^{\infty}\alpha(\lambda,n)|a_n|r^{n-1}\right\}$$

$$< \sum_{n=2}^{\infty}\alpha(\lambda,n)|a_n| - \mu(A-B)|b| - B\sum_{n=2}^{\infty}\alpha(\lambda,n)|a_n|$$

$$= \sum_{n=2}^{\infty}(1-B)\alpha(\lambda,n)|a_n| - \mu(A-B)|b|$$

$$\leq 0$$

Hence it follows that

$$\left|\frac{\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}{\mu(A-B)b - B\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}\right| < 1, z \in u$$

Therefore f belong to the class $V(\lambda+1,\mu,A,B,b)$ .We note that

$$f(z) = z - \frac{\mu(A-B)|b|z^n}{(A-B)\alpha(\lambda,n)}, n = 2,3...$$

is an external function with respect to above theorem, since for this function.

$$\left|\frac{\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}{\mu(A-B)b - B\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}\right| = 1$$

For $|z| = 1$, and the inequality is attend in (3.5.1).

In order to show that converse need to be true, we consider the function f given by(1.1.1) defined by

$$\frac{D^{\lambda+1}f(z)}{z} = \frac{1+\{B+\mu(A-B)b\}z}{1+Bz}$$

Where $-1 < B < 0, z \in u$ .Then it is easy to verify that

$$a_n = \frac{\mu(A-B)b(-B)^{n-2}}{\alpha(\lambda,n)}$$

But

$$\sum_{n=2}^{\infty}(1-B)\alpha(\lambda,n)|a_n|$$

$$> (A-B)|b|$$

Hence, the converses need not to be true.

4.6 MAXIMIZATION THEOREM:

THEOREM 4.6.1: If the function f is defined by (1.1.1) and it belong to the class $V(\lambda+1,\mu,A,B,b)$ .If δ is any complex number, then

$$\left|a_3-\delta a_2^2\right| \le \frac{\mu(A-B)|b|}{\alpha(\lambda,\beta)}\max\left\{1|d|\right\},$$

Where

$$d=\frac{B\left\{\alpha(\lambda,2)\right\}^2+\mu(A-B)b\delta\left\{\alpha,(\lambda,3)\right\}}{\left\{\alpha(\lambda,2)\right\}^2} \tag{4.6.1}$$

The inequality (4.6.1) is sharp.

PROOF: Since f belongs to the class $V(\lambda+1,\mu,A,B,b)$,

We have

$$1+\frac{1}{b}\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}=(1-\mu)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}, \tag{4.6.2}$$

Where

$w(z)=\sum_{k=1}^{\infty}C_k z^k$ is analytic in u and satisfy the conditions $w(0)=0,|w(z)|<1$

From (4.6.2) ,we have

$$w(z)=\frac{\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}{\mu(A-B)b-B\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}$$

$$=\frac{\sum_{n=2}^{\infty}\alpha(\lambda,n)a_n z^n}{\mu(a-b)-b\sum_{n=2}^{\infty}\alpha(\lambda,n)a_n z^n}$$

$$=\frac{1}{\mu(A-B)}\left[\alpha(\lambda,2)a_2 z+\alpha(\lambda,3)z^2+\frac{B\left\{\alpha(\lambda,2)\right\}^2+a^2z^2}{\mu(A-B)\lambda}\right]$$

And then comparing the coefficient of z and z^2 on the both sides, we have

$$c_1 = \frac{\alpha(\lambda,2)a_3}{\mu(A-B)b}$$

and

$$c_2 = \frac{\alpha(\lambda,3)a^3}{\mu(A-B)b} + \frac{B\{\alpha(\lambda,2)\}a_2^2}{\mu^2(A-B)b^2}$$

Thus

$$a_2 = \frac{\mu(A-B)bc_1}{\alpha(\lambda,2)}$$

and

$$a_3 = \frac{\mu(A-B)b(c_2 - Bc_1^2)}{\alpha(\lambda,3)}$$

Hence

$$a_3 - \delta a_2^2 = \frac{\mu(A-B)b}{\alpha(\lambda,3)}\left(c_2 - dc_i^2\right)$$

Where

$$d = \frac{B\{\alpha(\lambda,2)\}^2 + \mu(A-B)b\delta\{\alpha(\lambda,3)\}}{\{\alpha(\lambda,2)\}^2}$$

Therefore

$$\left|a_3 - \delta a_2^2\right| = \frac{\mu(A-B)|b|}{\alpha(\lambda,3)}\left|c_2 - c_1^2\right|$$

Using the Lemma (4.2.2) in the above equations, we get

$$\left|a_3 - \delta a_2^2\right| \le \frac{\mu(A-B)|b|}{\alpha(\lambda,3)}\max\{1,|d|\}.$$

Since the inequality (4.2.2) is sharp, so that the inequality (4.6.1) must also be sharp.

4.7 DISTORTIAN THEOREM:

THEOREM 4.7.1: If f belongs to class $V(\lambda+1,\mu,A,B,b)$

$$\operatorname{Re}\left\{\frac{D^{\lambda+1}f(z)}{z}\right\} \ge \frac{\left(1-B^2r^2\right) - \mu Br^2(A-B)\operatorname{Re}(b) - \mu(A-B)|b|r}{1-B^2r^2} \quad (4.7.1)$$

And

$$\operatorname{Re}\left\{\frac{D^{\lambda+1}f(z)}{z}\right\} \le \frac{\left(1-B^2r^2\right)-\mu Br^2\left(A-B\right)\operatorname{Re}(b)-\mu(A-B)|b|r}{1-B^2r^2} \qquad (4.7.2)$$

The inequalities (4.7.1) and (4.7.2) are sharp.

PROOF: Since f belongs to the class $V\left(\lambda+1,\mu,A,B,b\right)$,we have

$$1+\frac{1}{b}\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}=\left(1-\mu\right)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}=\left(1-\mu\right)+\mu p(z) \qquad (4.7.3)$$

Where

$$P(z)=\left\{1+Aw\left(z\right)\right\}/\left\{1+Bw\left(z\right)\right\}$$

It is well known that the transformation $P(z)=\left\{1+Aw\left(z\right)\right\}/\left\{1+Bw\left(z\right)\right\}$ maps the circle equations (4.7.2) and (4.7.4) yield

$$\left|P(z)-\left\{\frac{1-ABr^2}{1-B^2r^2}\right\}\right| \le \frac{\left(A-B\right)}{\left(1-B^2r^2\right)} \qquad (4.7.4)$$

$$\left|\frac{\mu+\frac{1}{b}\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}{\mu}-\left\{\frac{1-ABr^2}{1-B^2r^2}\right\}\right| \le \frac{\left(A-B\right)r}{\left(1-B^2r^2\right)}$$

Or

$$\left|\frac{D^{\lambda+1}f(z)}{z}-\frac{\left\{\left(1-B^2r^2\right)-b\mu Br^2\left(A-B\right)\right\}}{\left(1-B^2r^2\right)}\right| \le \frac{\mu\left(A-B\right)|b|r}{1-B^2r^2}$$

Hence

$$\operatorname{Re}\left\{\frac{D^{\lambda+1}f(z)}{z}\right\} \ge \frac{\left(1-B^2r^2\right)-\mu\left(A-B\right)Br^2\operatorname{Re}\left(b\right)-\mu\left(A-B\right)|b|r}{1-B^2r^2}$$

And

$$\operatorname{Re}\left\{\frac{D^{\lambda+1}f(z)}{z}\right\} \le \frac{\left(1-B^2r^2\right)-\mu\left(A-B\right)Br^2\operatorname{Re}\left(b\right)-\mu\left(A-B\right)|b|r}{1-B^2r^2}$$

By considering the functions f is defined by

$$\frac{D^{\lambda+1}f(z)}{z}=\frac{1+\left\{\mu\left(A-B\right)b+B\right\}}{1+Bze^{ir}},$$

Where, $e^{ir}=\dfrac{|b|-Bzb}{b-Bz|b|}$

We find that bounds in (4.7.1) and (4.7.2) are sharp at $z = e^{ir}$. respectively

4.8 OPERATOR:

THEOREM 4.8.1 : Let r be a real number such that $r > -1$. If f belongs to the Class $V(\lambda, \mu, A, B, b)$,then the functions F is defined by

$$F(z) = \frac{r+1}{z^r} \int_0^\infty t^{r-1} f(t) dt \tag{4.8.1}$$

Also belongs to $V(\lambda, \mu, A, B, b)$.

PROOF: From (4.8.1), it easy to verify that

$$z\left(D^{\lambda+1}F(z)\right)' = (r+1)D^{\lambda+1}F(z) - rD^{\lambda+1}F(z) \tag{4.8.2}$$

Suppose that

$$H_1(z) = \frac{1 + \{B + \mu(A-B)\}w(z)}{1 + Bw(z)} \tag{4.8.3}$$

Where

$$H_1(z) = 1 + \frac{1}{b}\left\{\frac{D^{\lambda+1}F(z)}{z} - 1\right\}$$

w(o)=0 and w is either analytic or meromorphic in u.

Differentiating (4.8.2) and using the identity (4.8.2),we get

$$\left[1 + \frac{1}{b}\left\{\frac{D^{\lambda+1}F(z)}{z} - 1\right\}\right] - m$$

$$= \frac{(1-m) + \{B + \mu(A-B) - Bm\}w(z)}{1 + Bwz} + \frac{\mu(A-B)}{(r-1)} \frac{zw'(z)}{\{1 + Bw(z)\}^2}$$

The required result can be obtained now from (4.8.4) by using the same technique as applied in (4.8.4) in proof of theorem (4.3.1).

4.9 CLOSURE THEOREM:

THEOREM 4.9.1 : If the function f and g belongs to the class $V(\lambda, \mu, A, B, b)$ and $0 \le S \le 1$ then the function F is given $F(z) = sf(z) + (1-s)g(z)$ also belongs to $V(\lambda, \mu, A, B, b)$.

PROOF: Since f and g belong to the class $V\left(\lambda,\mu,A,B,b\right)$, by Lemma (4.2.3), we have

$$\left|\left[1+\frac{1}{b}\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}\right]-m\right|<M$$

and

$$\left|\left[1+\frac{1}{b}\left\{\frac{D^{\lambda+1}g(z)}{z}-1\right\}\right]-m\right|<M, z\in u$$

Therefore

$$\left|\left[1+\frac{1}{b}\left\{\frac{D^{\lambda+1}F(z)}{z}-1\right\}\right]-m\right|$$

$$\left|\left[1+\frac{1}{b}\left\{\frac{sD^{\lambda+1}f(z)+\left(1-s\right)D^{\lambda+1}g(z)}{z}-1\right\}\right]-m\right|$$

$$=\left|S\left[1+\frac{1}{b}\left\{\frac{D^{\lambda+1}F(z)}{z}-1\right\}\right]-m\right|+(1-s)\left|\left[1+\frac{1}{b}\left\{\frac{D^{\lambda+1}g(z)}{z}-1\right\}\right]-m\right|$$

$$\leq\left|\left[1+\frac{1}{b}\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}\right]-m\right|+(1-s)\left|\left[1+\frac{1}{b}\left\{\frac{D^{\lambda+1}g(z)}{z}-1\right\}\right]-m\right|$$

$$<sM+(1-s)M=M$$

Hence $f\in V(\lambda,\mu,A,B,b)$.

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CHAPTER 5

CERTAIN SUB CLASSES OF ANALYTIC FUNCTIONS RELATED TO COMPLEX ORDER (II)

5.1 INTRODUCTION:

In this Chapter, we have also introduced another class by using the convolution techniques as follows. A function *f* of A belongs to the class $G(\lambda, \mu, A, B, b)$ if and only if there exists a function w belonging to the class H such that

$$1+\frac{1}{b}\left\{\frac{z^{\lambda}(D^{\lambda}f(z))'}{D^{\lambda}f(z)}-1\right\}=\left(1-\mu\right)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}, z\in u,$$

(5.1.1)

Where

$-1\leq B<A\leq 1, 0<\mu\leq 1, \lambda-1, D^{\lambda}f(z)$ is defined by(5.1.1) and b is any non zero complex number. Using the identity, we have

$z(D'f(z))'=\left(\lambda+1\right)D^{\lambda+1}f(z)-\lambda D^{\lambda}f(z)$ in(4.1.1). we have

$$1+\frac{\left(\lambda+1\right)}{b}\left\{\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\right\}\left(1-\mu\right)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}, z\in u,$$

(5.1.2)

It is easy to see that the conditions(5.1.1) and (5.1.2) are equivalent to

$$\left|\frac{\left\{\frac{z\left(D^{\gamma}f(z)\right)'}{D^{\gamma}f(z)}-1\right\}}{\mu(A-B)b-B\left\{\frac{z\left(D^{\gamma}f(z)\right)'}{D^{\gamma}f(z)}-1\right\}}\right|<1, z\in u, \tag{5.1.3}$$

and

$$\left|\frac{(\lambda+1)\left\{\frac{z\left(D^{\gamma}f(z)\right)'}{D^{\gamma}f(z)}-1\right\}}{\mu(A-B)b-B\left\{\frac{z\left(D^{\gamma}f(z)\right)'}{D^{\gamma}f(z)}-1\right\}}\right|<1, z\in u,$$

Respectively

This chapter is divided into six sections for the systematic study to the class $G(\lambda, \mu A, B, b)$ .Section 5.2 provided some lemma that are used in succeeding section of this chapter .In Section 5.3,we have obtained the coefficient estimate for section f belonging to the class $G(\lambda, \mu, A, B, b)$ .In section5.4,we have investigate the sufficient condition in terms of coefficient, for the functions f belonging to the class $G(\lambda, \mu, A, B, b)$.In sections 5.5, we have determined the maximization of $|a_3 - a_2^2|$ for the complex of over the class $G(\lambda, \mu, A, B, b)$.In section 5.6, we have found the necessary and sufficient condition in term of convolution for the function f belonging to class $G(\lambda, \mu, A, B, b)$.

5.2 PRELMINARY LEMMAS

In this section, we state the Lemma (5.2.1) due to Robertson and prove a Lemma (5.2.2), that are need in our investigations.

LEMMA 5.2.1 $h(z) = \sum_{p=q}^{\infty} |d_p|^2 z^p, H(z) = \sum_{p=q}^{\infty} |D_p|^2 z^p, q \geq 0$

If $h(z) = w(z)$, where $w(0) = 0$ and $d|w(z)| < 1$ in u, then dq=0 , then we have,

$$\sum_{p=q+1}^{k} |d_p|^2 \leq \sum_{p=q}^{k-1} |D_p|^2, (k = q+1, q+2, ..)$$

LEMMA5.2.2 : For a fixed integer $k, k \geq 3$,

$$\text{Let} \quad M_j = \frac{|\mu(A-B)b - (j-2)B|^2}{(\lambda + j - 1)^2}, j = 2,3.....k \tag{5.2.1}$$

And

$$c(\lambda, p) + \frac{(\lambda+1)_{p-1}}{(p-1)!}$$

$$=\frac{(\lambda+1)(\lambda+2)..(\lambda+p-1)}{(p-1)!},(P=2,3...)$$

Then

$$\frac{1}{\{(k-1)c(\lambda,k)\}^2}\left[\mu^2(A-B)^2|b|^2\right]+\sum_{p=2}^{k-1}\left\{\left|\mu(A-B)b-(p-1)B^2\right|\right\}.$$

$$.\{(\lambda,k)\}^2\prod_{j=2}^{p}M_j\Bigg]=\prod_{j=2}^{p}M_j$$

PROOF: We shall prove (5.2.2) by mathematical inductions on k. A brief calculation to show that (5.2.2) holds for k=3**.** We assume that (5.2.2**)** is valid for k=4, 5…t-1**,** then for k=t

, the left side of (5.2.2) gives.

$$\frac{1}{\{(t-1)c(\lambda,t)\}^2}\left[\mu(A-B)|b|^2\right]+$$

$$+\sum_{p=1}^{t-1}\left\{\left|\mu(A-B)b-(p-1)B\right|-(p-1)^2\right\}\{c(\lambda,k)\}^2\prod_{j=2}^{p}M_j\Bigg]$$

$$=\frac{1}{\left\{(t-1)c(\lambda,t)^2\right\}}\left[\begin{array}{l}\mu^2(A-B)^2|b|^2+\sum_{p=2}^{t-2}\left\{\left|\mu(A-B)b-(p-1)B\right|^2\right\}\{c(\lambda,p)\}^2\prod_{j=2}^{p}M_j\\+\left\{\left|\mu(A-B)b-(t-2)B\right|^2-(t-2)^2\right\}\{c(\lambda,t-1)\}^2\prod_{j=2}^{t-1}M_j\end{array}\right]$$

$$=\frac{1}{\{(t-1)c(\lambda-t)\}^2}\left[\begin{array}{l}\{(t-2)c(\lambda,t-1)\}^2\prod_{j=2}^{t-1}M_j+\\\{\lambda+t-1\}^2M_t-(t-2)\}\{c(\lambda,t-1)\}\prod_{j=2}^{t-1}M_j{}^2\end{array}\right]=\prod_{j=2}^{t}M_j$$

This concludes the proof of (5.2.2)

5.3 COEFFICIENT ESTIMATES:

THEOREM5.3.1 : Let the function f is defined by (1.1.1) belongs to the class $G(\lambda,\mu,A,B,b)$.

Then

$$|a_n|\le\frac{\mu(A-B)|b|}{(\lambda+1)} \quad (5.3.1)$$

And if $\left|\mu(A-B)b-B\right|\le 1$ an $n\ge 3$

Then,

$$|a_n| \le \frac{\mu(A-B)|b|(n-2)}{(\lambda+1)_{n-1}} \tag{5.3.2}$$

Furthermore, if

$$|\mu(A-B)b-(n-2)B| > (n-2), n \ge 3,$$

Let $$M = \left[\frac{|\mu(A-B)b-(n-2)B|}{(n+2)} \right]$$

be the greatest integer is less than or equal to the expression with in the square bracket.

Then

$$|a_n| \le \frac{1}{(\lambda+1)_{n-1}} \prod_{j=2}^{n} |\mu(A-B)b-(j-2)B| \text{ for } n = 3,4,...M+2; \quad \textbf{(5.3.3)}$$

And

$$|a_n| \le \frac{(n-2)!}{(m+2)!(\lambda+1)_{n-1}} \prod_{j=2}^{m+3} |\mu(A-B)b-(j-2)B| \text{ form >M+ 1} \quad \textbf{(5.3.4)}$$

The bounds (5.3.1) and (5.3.3) are sharp for all admissible μ, A, B, b, λ and for each n.

PROOF: From (5.1.1), we have,

$$z\left(D^{\lambda} f(z)\right)' - D^{\lambda} f(z) = \left[\left\{\mu(A-B)b+B\right\} D^{\lambda} f(z) - Bz\left(D^{\lambda} f(z)'\right)\right] w(z)$$

(5.3.5)

Since

$$D^{\lambda} f(z) = z + \sum_{p=2}^{\infty} c(\lambda, p) a_p z^p$$

It follows that (5.3.5) is equivalent to

$$\sum_{p=2}^{\infty} (p-1) c(\lambda, p) a_p z^p = \left[\sum_{p=1}^{\infty} \left\{ \mu(A-B)b-(p-1)B \right\} c(\lambda, p) a_p z^p \right] w(z),$$

Where $a_1 = 1$. Using the Lemma (5.2.1), we have

$$\sum_{p=2}^{n}(p-1)^2\{c(\lambda,p)\}^2|a_p|^2 \le \sum_{p=1}^{n-1}|\mu(A-B)b-(p-1)B|\{c(\lambda,p)\}^2|a_p|^2$$

This simplifies to

$$|a_n|^2 \le \frac{1}{\{(n-1)c(\lambda,n)\}^2}\left[\begin{array}{l}\mu^2(A-B)^2|b|^2+\\ \sum_{p=2}^{n-1}\left\{|\mu(A-B)b-(p-1)B|^2-(p-1)^2\right\}\{c(\lambda,p)\}^2|a_p|^2\end{array}\right]$$

(5.3.6)

For every n=2, 3…

Or

n=2, we have

$$|a_2|^2 \le \left\{\frac{\mu(A-B)|b|^2}{(\lambda+1)}\right\}$$

This proves that (5.3.1)

Suppose that

$|\mu(A-B)b-(n-2)B| \le (n-2)$ and $n \ge 3$ .Then it follows that

$|\mu(A-B)b-(n-2)B| \le (n-2)$ and $n \ge 3$

Since all the terms under the summations in (5.3.6) are non positive, we obtain

$$|a_n| \le \frac{\mu(A-B)|b|}{(n-1)c(\lambda,n)}, n \ge 3$$

This gives (5.3.2). However, if

$$|\mu(A-B)b-(n-2)B| > (n-2), n \ge 3$$

Then all the terms under the summation in (5.3.6) are positive. We shall established for $n > 3$ and n<M+2 from by mathematical inductions .For n=3, we have,

$$|a_3|^2 \le \left[\frac{\mu(A-B)[b]|\mu(A-B)b-B|}{(\lambda+1)(\lambda+2)}\right],$$

Which proves (5.3.3) hold (5.3.3) holds for n=4,5….k-1,then n=k,(4.3.6) yields

$$|a_k|^2 \le \frac{1}{\{(k+1)c(\lambda,k)\}^2}\left[\mu(A-B)^2|b|\right]+$$

$$+\sum_{p=2}^{k-1}\left\{|\mu(A-B)b-(p-1)B|^2-(p-1)\right\}\{c(\lambda,p)\}^p|a_p|^2$$

$$\le \frac{1}{\{(k-1)c(\lambda,k)\}^2}\left[\mu^2(A-B)|b|^2+\sum_{p=2}^{k-1}\left\{|\mu(A-B)b-(p-1)B|^2-(p-1)^2\right\}\{c(\lambda,p)\}^2\right]$$

$$.\prod_{j=2}^{p}\frac{|\mu(A-B)b-(j-2)B|^2}{(\lambda+j-1)^2}$$

$$=\prod_{j=2}^{k}\frac{|\mu(A-B)b-(j-2)B|^2}{(\lambda+j-1)^2}$$

By Lemma (5.2.2) .It is now that easy to show that (5.3.3) holds for $n \ge M+2$. Finally, Suppose n>M+2. Then we may write (5.3.6)

$$|a_n|^2 \le \frac{1}{\{(n-1)c(\lambda,n)\}^2}\left[\mu\left(A-B)^2\right)|b|^2+\right.$$

$$+\sum_{p=2}^{m+2}\left\{|\mu(A-B)b-(p-1)B|^2-(p-1)\right\}\{c(\lambda,p)\}^2|a_p|^2$$

$$+\sum_{p=m+3}^{n-1}\left\{|\mu(A-B)b-(p-1)B|^2-(p-1)\right\}\{c(\lambda,p)\}^2|a_p|^2$$

$$\le \frac{1}{\{(n-1)c(\lambda,n)\}^2}\left[\mu\left(A-B)^2\right)|b|^2+\right.$$

$$\le \frac{1}{\{(n-1)c(\lambda,n)\}^2}\left[\mu\left(A-B)^2\right)|b|^2+\sum_{p=2}^{m+2}\left\{|\mu(A-B)b-(p-1)B|^2-(p-1)\right\}\{c(\lambda,p)\}^2|a_p|^2.\right.$$

$$\left.\prod_{j=2}^{p}\frac{|\mu(A-B)b-(j-2)B|^2}{(\lambda+j-1)^2}\right]$$

$$=\left[\frac{(m+2)c(\lambda,m+3)}{(n-1)c(\lambda,n)}\right]\sum_{j=2}^{m+3}\frac{\left||\mu(A-B)b-(j-2)B|^2\right|}{(\lambda+j-1)}$$

$$\left[\frac{(n-2)!}{(m+1)!(\lambda+1)(\lambda+2)......(\lambda+n-1)}\right]^2.\sum_{j=2}^{m+3}|\mu(A-B)b-(j-2)B|^2$$

$$+\sum_{p=2}^{m+2}\left\{|\mu(A-B)b-(p-1)B|^2-(p-1)\right\}\{c(\lambda,p)\}^2|a_p|^2$$

By the application of Lemma (5.2.2) and (5.3.3) follows from above. For the function $f_n(z)$ is given by and using the convolution technique we have,

$$f_n * \frac{z}{(1-z)^{\lambda+1}} = \begin{cases} z(1+Bz)\mu(A-B)b/B(n-1), B \neq 0 \\ z\exp\{\mu(A-B)+B\}bz^{n-1}/n-1, B=0 \end{cases}$$

Where $|\mu(A-B)b-B|<1.$ Finally, the inequality (5.3.3) is sharp for the function $f(z)$ is given by

$$f_n * \frac{z}{(1-z)^{\lambda+1}} = \begin{cases} z(1+Bz)\mu(A-B)b/B, B \neq 0 \\ [z\exp\{\mu(A-B)+B\}bz], B=0 \end{cases}$$

Where

$$|\mu(A-B)b-(n-2)B|>(n-2), n\geq 3.$$

REMARK: If we take $\mu=1$.

5.4 SUFFICIENT CONDITION:

THEOREM 5.4.1: Let f is defined by (1.1.1) be analytic in U. If

$$\sum_{n=2}^{\infty}\{(n-1)+|\mu(A-B)b-(n-1)|B\}c(\lambda,n)|a_n| \leq \mu(A-B)|b|,$$

(5.4.1)

Holds for some $\lambda \geq 0,$ then the functions f belongs to the class $G(\lambda,\mu,A,B,b)$

PROOF: Suppose that the inequality (5.4.1) holds. Then, for $z \in u$,we have

$$|z(D^{\lambda}f(z))'-D^{\lambda}f(z)|-|\{\mu(A-B)b+B\}D^{\lambda}f(z)B(D^{\lambda}f(z))'|$$

$$\left|\sum_{n=2}^{\infty}(n-1)c(\lambda,n)a_n z^n\right| - \left|\mu(A-B)bz + \sum_{n=2}^{\infty}\left\{\mu(A-B)b-(n-1)B\right\}c(\lambda,n)a_n z^n\right|$$

$$\leq \left[\sum_{n=2}^{\infty}(n-1)c(\lambda,n)|a_n|r^n - \mu(A-B)|b|r^n\right] - \left[\sum_{n=2}^{\infty}\left\{|\mu(A-B)b-(n-1)B|\right\}c(\lambda,n)|a_n|r^n\right]$$

$$< \sum_{n=2}^{\infty}(n-1)c(\lambda,n)|a_n| - \mu(A-B)|b| + \sum_{n=2}^{\infty}\left\{|\mu(A-B)b-(n-1)B|\right\}c(\lambda,n)|a_n|$$

$$= \sum_{n=2}^{\infty}\left\{|(n-1)+\mu(A-B)b-(n-1)B|\right\}c(\lambda,n)|a_n| - \mu(A-B)|b|$$

$\leq 0,$ by the inequality (5.4.1).

REMARK: If we take $\mu=1, A=1$ B=-1. Theorem (5.4.1) coincides with the corresponding result of Chaudhary.

5.5 MAXIMIZATION THEOREM

5.5.1 THEOREM If f is defined by(1.1.1) belongs to the class $G(\lambda,\mu,A,B,b)$ and δ is any complex number, then

$$\left|a_3-\delta a_2^2\right| < \frac{\mu(A-B)|b|}{2c(\lambda,3)}\max\left\{1,|d|\right\}, \qquad (5.5.1)$$

Where

$$d = \frac{2\delta\mu(A-B)bc(\lambda,3) - \left\{\mu(A-B)b-B\right\}\left\{c(\lambda,2)\right\}^2}{\left\{c(\lambda,2)\right\}^2}$$

The inequality (5.5.1) is sharp for each δ .

PROOF: Since $f \in G(\lambda,\mu,A,B,b)$,then we have

$$1+\frac{1}{b}\left\{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}-1\right\} = (1-\mu)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}, z\in u$$

(5.5.2)

Where $w(z)=\sum_{k=1}^{\infty}c_k z^k$ from (5.5.2), we have

$$w(z) = \frac{z(D^\lambda f(z))' - D^\lambda f(z)}{\left\{\mu(A-B)b+B\right\}D^\lambda f(z) - Bz(D^\lambda f(z))'}$$

$$w(z)=\frac{\sum_{n=2}^{\infty}(n-1)c(\lambda,n)a_n z^n}{\mu(A-B)bz+\sum_{n=2}^{\infty}\{\mu(A-B)b-(n-1)B\}c(\lambda,n)a_n z^n}$$

Or

$$w(z)=\frac{1}{\mu(A-B)}\left[c(\lambda,2)a_2 z+2c(\lambda,3)a_3 z^2-\frac{\{\mu(A-B)b-B\}}{\mu(A-B)b}.\{c(\lambda,2)\}^2 a_2 z^2+...\right]$$

Equating the coefficients of z and z^2 on the both sides, we get,

$$a_2=\frac{\mu(A-B)bc_2}{c(\lambda,2)}$$

And

$$a_3=\frac{\mu(A-B)bc_2+\mu(A-B)b\{(A-B)-B\}c_1^2}{2c(\lambda,3)}$$

Thus, we

$$a_3-\delta a_2^2=\frac{\mu(A-B)b}{2c}\left(c_2-dc_1^2\right),$$

Where

$$d=\frac{2\delta\mu(A-B)c(\lambda,3)-\{\mu(A-B)b-b\}\{c(\lambda,2)\}^2}{\{c(\lambda,2)\}^2}$$

Hence

$$\left|a_3-\delta a_2^2\right|=\frac{\mu(A-B)|b|}{2c(\lambda,3)}\max\{1,|d|\},$$

Since the inequality (5.2.2) is sharp, so that the inequality(4.5.1) must also be sharp.

REMARK: If we take $\mu=1, A=1$ and B=-1, theorem (5.5.1) coincides with corresponding result

5.6 CONVOLUTION CONDITION:

THEOREM 5.6.1 : A function f belong to the class $G(\lambda,\mu,A,B,b)$

$$f(z)*\left[\frac{-\mu(A-B)bzx+\left[(\lambda+1)+X\{\mu(A-B)b+(\lambda+1)\}B\right]z^2}{(1-z)^{\lambda+2}}\right]$$

$\neq 0$

(5.6.1)

In where, $0<|z|<1,\ |X|=1$ and $X=1$

PROOF: Let the function f belongs to the class $G(\lambda,\mu,A,B,b)$ then

$$1+\frac{1}{b}\left\{\frac{z\left(D^{\lambda}f(z)\right)'}{D^{\lambda}f(z)}-1\right\}\neq(1-\mu)+\mu\left\{\frac{1+Ax(z)}{1+Bx(z)}\right\}$$

(5.6.2)

$|X|=1$ and $X=1$ in $o<|z|<1,$ is equivalent to

$$(1+Bx)\left\{z(D^{\lambda}f(z))'-D^{\lambda}f(z)\right\}-b\mu(A-B)XD^{\lambda}f(z)\neq 0 in|z|>1 \qquad (5.6.3)$$

We know that

$$z(D^{\lambda}f(z))'=(\lambda+1)D^{\lambda+1}f(z)-\lambda D^{\lambda}f(z)$$

(5.6.4)

Using (5.6.4) in (5.6.3) , we have,

$$(1+Bx)\left\{(\lambda+1)D^{\lambda+1}f(z)-\lambda D^{\lambda}f(z)\right\} -$$
$$\{1+Bx+b\mu(A-B)\}D^{\lambda}f(z)\neq 0 \text{ in } 0>|z|<1 \qquad (5.6.5)$$

Since $\qquad D^{\lambda}f(z)=\dfrac{z}{(1-z)^{\lambda+1}}*1$

reduce to $\qquad f(z)*\left[\dfrac{-\mu(A-B)bzx+\left[(\lambda+1)+X\{\mu(A-B)b+(\lambda+1)B\}\right]z^2}{(1-z)^{\lambda+2}}\right]$

Which is required convolution condition. The converse part follows easily since all the steps can be retracted back.

REMARK: If we take μ=1,A=1and B=-1,theorem (5.6.1) coincides with the corresponding result

CHAPTER 6

ANALYTIC FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVE(I)

6.1 INTRODUCTION:

Let $J(P)$ dente subclass of $Y(P)$ consisting of analytic functions and p-valent functions which can be expressed in the form.

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}, \, p \in n$$

(6.1.1)

If f and g are any two functions in the class $J(p)$ such that f is defined by(6.1.1)and

$$g(z) = z^p - \sum_{n=1}^{\infty} |b_{p+n}| z^{p+n}, \, p \in n \quad (6.1.2)$$

Then the Modified Hadamard Product of f and g, denoted by is defined by the power series

$$(f * g)(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}||b_{n+p}| z^{p+n} \quad (6.1.3)$$

Now, we have made known the class $J(A, B, p, \delta)$ of Analytic Functions in terms of Fractional Derivatives Operator as defined below. A function of $J(P)$ belongs to the class $J(A, B, p, \delta)$.If and only if there exists a functions w belongs to the class x such that.

$$\Omega_z^{(\delta,p)} f(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \qquad \text{where} 1 \le A < B \le 1$$

(6.1.4)

And

$$\Omega_z^{(\delta,p)} f(z) = \frac{\Gamma(p - \delta + 1)}{\Gamma(p + 1)} z^{\delta - p} D_z^{\delta} f(z)$$

(6.1.5)

Here $D_z^{\delta} f(z)$ denotes the Fractional Derivative of f(z) order and is defined by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\varsigma) d\varsigma}{(z - \varsigma)\delta} \quad , \quad \text{Where } 0 \le \delta < 1 \quad (6.1.6)$$

Where $0 \le \delta < 1$ and f is Analytic Functions in a simply connected regions of z-plane. The origin multiplicity of $(z-\varsigma)$ is removed by $\log(z-\varsigma)$ to be real, when
$z-\varsigma > 0$
With $D_z^1 f(z) = f'(z)$
The condition (6.1.4) is equivalent to

$$\left| \frac{\Omega_z^{(\delta,p)} f(z) - 1}{B\Omega^{(\delta,p)} f(z) - A} \right| < 1, z \in u$$

(6.1.7)

By the specific value to A,B,p and δ from(6.1.7), we obtain the following important subclasses studied by researcher in earlier works.

[i]For A=2a-1 and B=1,we obtain the class of functions f satisfying the condition.

$$\left| \frac{\Omega_z^{(\delta,p)} f(z) - 1}{\Omega^{(\delta,p)} f(z) - 2a + 1} \right| < \beta, z \in u$$

[ii]For , $\delta = 0$,we obtain the class of functions f is satisfying the condition.

$$\left| \frac{z^{-p} f(z) - 1}{Bz^{-p} f(z) - A} \right| < 1, z \in u$$

In the sections 6.2 , we have obtained the necessary and sufficient condition in terms coefficients for the functions f belonging for the class $J(A,B,p,\delta)$ .In section 6.3, we have investigated the Distortion Properties for the class $J(A,B,p,\delta)$.In sections 6.4 ,we have studied Integral Operator of the form.

$$F(z) = \frac{c+p}{z^c} \int_0^z tc - 1 f(t) dt$$

(6.1.8)

When $f \in J(A,B,p,\delta)$ .In section 6.5,we have found the radius of p-valent Starlikeness for the class $J(A,B,p,\delta)$. In section 6.6, we have obtained the radius of p-valent convexity for the class $J(A,B,p,\delta)$. In section 6.7,we have obtained the results involving Modified Hadamard Product of two functions belonging to the class $J(A,B,p,\delta)$. In sections 6.8,we have obtained the contentment relations related to the class $J(A,B,p,\delta)$.In sections 6.9, we have investigated some Closure Properties for the class $J(A,B,p,\delta)$.

Note- In this Chapter, we assume that

$$\phi(n,p,\delta)=\frac{\Gamma(n+1+p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+1+p-\delta)}$$

6.2 NECESSARY AND SUFFICIENT CONDITIONS:

THEOREM 6.2.1 : A function f is defined by(6.1.1) is in the class $J(A,B,p,\delta)$ .If and only if

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)\left|a_{p+n}\right|\leq(B-A)$$

The inequality (6.2.1)is sharp.

PROOF: Let$|z|=1$. Then

$$\left|\Omega_z^{(\delta,p)}f(z)-1\right|-\left|B\Omega_z^{(\delta,p)}f(z)-A\right|$$

$$=\left|n\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|\left|z^n\right|\right|-\left|(B-A)-B\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|\left|z^n\right|\right|\leq 0,$$ by the

$$\leq\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)\left|a_{p+n}\right|-(B-A)$$

hypothesis.

Hence, by the Maximum Modulus Theorem $f\in J(A,B,p,\delta)$.

To prove the converts,

Let

$$\left|\frac{\Omega_z^{(\delta,p)}f(z)-1}{B\Omega_z^{(\delta,p)}-A}\right|=\left|\frac{-\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|z^n}{(B-A)-B\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|z^n}\right|<1, z\in u$$

Since$|\text{Re}(z)|\leq|z|$ for all z.

We have

$$\text{Re}\left[\frac{\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|z^n}{(B-A)-B\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|z^n}\right]<1$$

Choose the value of z on the real axis so that$\Omega_z^{(\delta,p)}f(z)$is real. Upon the clearing the denominotor in (6.2.2) and letting $z\to 1$ through the real value, we have

This is the complete proof the theorem.

The function

$$f(z)=z^{p}-\frac{(B-A)}{(1+B)\phi(n,p,\delta)}z^{n},n\in N$$

This function is an external function.

COROLLARY 6.2.1 Let the function f defined by (5.1.1) belongs to class $J(A,B,p,\delta)$.Then

$$\left|a_{p+n}\right|\leq\frac{(B-A)}{(1+B)\phi(n,p,\delta)},$$ for every integer , $n\in N$

6.3 DISTORTIAN THEOREM:

THEOREM 6.3.1 : $f\in j(A,B,p,\delta)$ .Then

$$|z|^{p}-\frac{(B-A)(1+p-\delta)}{(1+B)(1+p)}|z|^{p+1}\leq|f(z)|\leq|z|^{p}-\frac{(B-A)(1+p-\delta)}{(1+B)(1+p)}|z|^{p+1}$$

(6.3.1)

And

$$\frac{\Gamma(1+p)}{\Gamma(1+p-\delta)}|z|^{p-\delta}-\frac{(B-A)\Gamma(1+p)}{(1+B)\Gamma(1+p-\delta)}|z|^{p-\delta+1}\leq\left|D_{z}^{\delta}f(z)\right|$$

$$\leq\frac{\Gamma(1+p)}{\Gamma(1+p-\delta)}|z|^{p-\delta}-\frac{(B-A)\Gamma(1+p)}{(1+B)\Gamma(1+p-\delta)}|z|^{p-\delta+1}$$

(6.3.2)

Where z∈U

PROOF: Since

$$\frac{(1+p)(1+B)}{(1+p-\delta)}\sum_{n=1}^{\infty}\left|a_{p+n}\right|\leq\sum_{n=1}^{\infty}\phi(n,\phi,\delta)(1+B),\left|a_{p+n}\right|\leq(B-A)$$

It is evidently yields

$$\sum_{n=1}^{\infty}\left|a_{p+n}\right|\leq\frac{(B-A)(1+p-\delta)}{(1+B)(1+p)}$$

Consequently, we obtain

$$|f(z)|\geq|z|^{p}-|z|^{p+1}\sum_{n=1}^{\infty}\left|a_{p+n}\right|$$

$$\leq|z|^{p}-\frac{(B-A)(1+p-\delta)}{(1+B)(1+p)}|z|^{p+1}$$

$$|f(z)|\geq|z|^{p}+|p|^{p+1}\sum_{n=1}^{\infty}\left|a_{p+n}\right|$$

$$\leq|z|^{p}+\frac{(B-A)(1+p-\delta)}{(1+B)(1+p)}|z|^{p+1}$$

This proves the inequality (6.3.1)

Next, by using the second inequality (6.3.3), we observe that

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|\le\frac{(B-A)}{(1-B)}$$

Now

$$\left|z^p\Omega_z^{(\delta,p)}f(z)\right|\ge|z|^p-\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right||z|^{p+n}$$

$$\ge|z|^p-|z|^{p+1}\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|$$

$$\ge|z|^p-\frac{(B-A)}{(1-B)}|z|^{p+1}$$

And

$$\left|z^p\Omega_z^{(\delta,p)}f(z)\right|\le|z|^p+\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right||z|^{p+n}$$

$$\le|z|^p+|z|^{p+n}\sum_{n=1}^{\infty}\phi(n,p,\delta)\left|a_{p+n}\right|$$

$$\le|z|^p+\frac{(B-A)}{(1-B)}|z|^{p-1}$$

This has given the inequality (6.3.2).

6.3.1 COROLLARY: Under the hypothesis of Theorem (6.3.1), $f|z|$ is included in the disc with the center at the origin and radius r is given by

$$\Gamma=1+\frac{(B-A)(1+p-\delta)}{(1+B)(1+p)}$$

And $D_z^{\delta}f(z)$ is included in disc with its centre at the origin and radius R is given by

$$R=\frac{\Gamma(1+p)}{\Gamma(1+p-\delta)}\left\{1+\frac{(B-A)}{(1-B)}\right\}$$

6.4 INTEGRAL OPERATOR:

THEOREM 6.4.1 Let $c>-p$ if $f\in J(A,B,p,\delta)$ .Then the functions defined by (5.1.8) also belong to $J(A,B,p,\delta)$.

PROOF: From the definitions of (5.1.8) and (5.1.1), it easily proof that

$$F(z)=z^p-\sum_{n=1}^{\infty}\left|n_{p+n}\right|z^{p+n}$$

Where

$$\left|n_{p+n}\right| = \frac{(c+p)}{(c+p+n)}\left|a_{p+n}\right|$$

Therefore

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)\left|n_{p+n}\right|$$

$$= \sum_{n=1}^{\infty}\phi(n,p,\delta)\frac{(c+p)}{(c+p+n)}\left|a_{p+n}\right|$$

$$< \sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)\left|a_{p+n}\right|$$

$$\leq (B-A)$$

Hence, by theorem (6.2.1) $F \in J(A,B,p,\delta)$.

COROLLARY6.4.1 : If $f \in J(A,B,p,\delta)$.

Then

$$F(z) = z^{p-1}\int_0^z \frac{f(t)}{t^p} \quad \text{at} \quad f \in J(A,B,p,\delta)$$

THEOREM 6.4.2 : Let c>-p. Also let F be the class $J(A,B,p,\delta)$.Then the functions f is given by p-valent in the unit disc $|z| < R$,where

$$R = \begin{matrix}\inf \\ n \in r\end{matrix}\left[\frac{(1+B)(c+b)\Gamma(n+p)\Gamma(1+p-\delta)}{(B-A)(c+p+n)(n+1+p-\delta)}\right]$$

The result is sharp.

PROOF: Let $F(z) = z^p - \sum_{n=1}^{\infty}\left|a_{p+n}\right| z^{p+n} \in J(A,B,p,\delta)$

Then from (6. 1.8) it follows that

$$= z^p - \sum_{n=1}^{\infty}\left\{\frac{c+p+n}{c+p}\right\}\left|a_{p+n}\right| z^{p+n}$$

In order to established to required result, then we get,

$$\left|\frac{f'(z)}{z^p} - p\right| \leq p for\left|z < R\right|$$

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{n=1}^{\infty}(p+n)\left\{\frac{(c+p+n)}{c+p}\right\}\left|a_{p+n}\right| z^n$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p$$

$$\sum_{n=1}^{\infty}(p+n)\left\{\frac{(c+p+n)}{(c+p)}\right\}|a_{p+n}|z^{p} < p$$

(6.4.1)

But from the theorem (6.2.1),We obtain

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)\frac{(1+B)p}{(B-A)}|a_{p+n}| \le p$$

The inequality (6.4.1)will be satisfied if

$$\frac{(p+n)(c+p+n)}{(c+p+n)}|a_{p+n}||z^{n}| < \phi(n,p,\delta)\frac{(1+B)}{(B-A)}|a_{p+n}|$$

For each $n \in N$ or if

$$|z| < \left|\frac{(1+B)(c+p)\Gamma(p+n)\Gamma(1+p-\delta)}{(B-A)(c+p+n)\Gamma(p)\Gamma(n+1+p-\delta)}\right|^{\frac{1}{n}}$$

for each $n \in N$. Hence f is p –valent $|z| < R$.

Sharpness follows,if we take

$$F(z) = z^{p} - \frac{(B-A)\Gamma(1+p)\Gamma(n+1+p-\delta)}{(1+B)\Gamma(n+1+p)\Gamma(1+p-\delta)}$$

6.5: RADIUS OF P-VALENT STARLIKENESS:

THEOREM 6.5.1 : If we take $f \in J(A,B,p,\delta)$.Then f is p-valent starlike of order $(0 \le a < p)$ in the unit disc $|z| < R_1^*$,where $R_1^* = R_1^*(A,B,p,\delta)$

.

$$\inf_{n \in N}\left[\frac{(1+B)(p-a)\Gamma(n+1+p)\Gamma(1+p-\delta)}{(B-A)(n+p-a)\Gamma(1+p)\Gamma(n+1+p-\delta)}\right]^{\frac{1}{n}}$$

PROOF: In order to obtain the result, it is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| < (p-a)\, for\, |z| < R_1^*$$

Let

$$f(z) = z^{p} - \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n} .$$

Then we have

$$\sum_{n=1}^{\infty}\frac{(n+p-a)}{(p-a)}|a_{p+n}||z|^n<1$$

(6.5.1)

But from the theorem (6.2.1), we have

$$\sum_{n=1}^{\infty}\frac{(1+B)}{(B-A)}\phi(n,p,\delta)|a_{p+n}|\leq 1$$

Hence (6.5.1) will be satisfied if

$$\frac{(n+p-a)}{(p-a)}|z|^n<\frac{(1+B)}{(B-A)}\phi(n,p,\delta)$$

For each $n\in N$

Or if $$|z|<\left[\frac{(1+B)}{(B-A)}\frac{(p-a)}{(n+p-a)}\frac{\Gamma(n+1-p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+1+p-\delta)}\right]^{\frac{1}{n}}$$

For each $n\in N$

The result is sharp for the function

$$f(z)=z^p-\frac{(B-A)}{(1+B)}\frac{\Gamma(1+p)\Gamma(n+1+p-\delta)}{\Gamma(n+1+p)\Gamma(1+p-\delta)},n\in N$$

6.6 RADIUS OF P-VALENT CONVEXITY:

THEOREM 6.6.1: If $f\in J(A,B,p,\delta)$,then f is p-valent convex of order $(0\leq a<p)$ in the unit disc $|z|<R_2^*$ where $R_2^*=R_2^*(p,A,B,\delta,a)$

The result is sharp.

PROOF: In order to established that it is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<(p-a) for |z|<R_2^*$$

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|\leq\frac{\sum_{n=1}^{\infty}(p+n)n|a_{p+n}||z|^n}{p-\sum_{n=1}^{\infty}|a_{p+n}||z|^n}$$

Therefore

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|<(p-a)$$

$$\sum_{n=1}^{\infty}\frac{(p+n)(p+n+a)}{p(p-a)}|a_{p+n}||z|^n<1$$

But the theorem (6.2.1), we have

$$\sum_{n=1}^{\infty}a(n,p,\delta)\frac{(1+B)}{(B-A)}|a_{p+n}|\leq 1$$

Hence $(6.6.1)$ will be it is satisfied if

$$\frac{(p+n)(p+n+a)}{p(p-a)}|a_{p+n}||z|^n$$

$$<\frac{(1+B)}{(B-A)}a(n,p,\delta)|a_{p+n}| \text{For each } n\in N$$

or, if

$$|z|<\left[\frac{(1+B)(p-a)\Gamma(n+p)\Gamma(1+p-\delta)}{(B-A)(n+p-\delta)\Gamma(p)\Gamma(n+p+1-\delta)}\right], n\in N$$

Therefore f is convex in $|z|<R_2^*$.

The result is sharp with the external functions of the form

$$f(z)=z^p-\frac{(B-A)}{(1+B)}\frac{\Gamma(1+p)\Gamma(n+1+p-\delta)}{\Gamma(n+1-p)\Gamma(1+p-\delta)}z^{p+n}$$

6.7 SOME RESULTS INVOLVING MODIFIED PRODUCT:

THEOREM 6.7.1: Let the function $f_j(z)$,where $j=1,2,3,....m$ is defined by

$$f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n,j}|z^{p+n} \quad (j=1,2,3,....m)$$

Be the classes $J(A_j,B_j p,\delta)$ where $j=1,2,3,....m$, respectively.

Also let

$$\left\{\frac{\delta}{1+p}\right\}+\min_{1\leq j\leq m}\{B_j\}\geq 0.$$

Then

$$(f_1 * f_2....f_m)(z)\in J\left(\sum_{j=1}^{\infty}A_j\sum_{j=1}^{\infty}B_{j,p,\delta}\right) \tag{6.7.1}$$

PROOF: Since $f_j\in J(A_j,B_j,p,\delta)$ where $j=1,2,3,....m$ using by the theorem (6.2.1), we have

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B_j)|a_{p+n,j}|\leq(B_j-A_j) \tag{6.7.2}$$

And

$$\sum_{n=1}^{\infty}\left|a_{p+n,j}\right| \le \frac{\left(B_j - A_j\right)}{1+B_j}\frac{\left(1+p-\delta\right)}{\left(1+p\right)}, \quad \text{each } j = 1,2,3,....m \tag{6.7.3}$$

Using for any J_0 and (6.7.3) for the rest, we obtain

$$\sum_{n=1}^{\infty}\phi\left(n,p,\delta\right)\left[1+\prod_{j=1}^{m}B_j\right]\prod_{j=1}^{m}\left|a_{p+n,j}\right|$$

$$\le \frac{\left\{\dfrac{1+P-\delta}{1+p}\right\}\prod\limits_{j=1}^{m}\left(B_j - A_j\right)}{\prod\limits_{j=1, j\ne j_0}^{m}\left(1+B_j\right)}$$

$$\le \frac{\left[1-\dfrac{\delta}{\left(1+p\right)}\right]^{m-1}\left[\prod\limits_{j=1}^{m}B_j - \prod\limits_{j=1}^{m}A_j\right]}{\left[1+\min\limits_{1\le j\le m}\left[B_j\right]\right]^{m-1}}$$

$$\le \prod_{j=1}^{m}B_j - \overset{m}{\underset{j=1}{A_j}}$$

$$0 < \left[\frac{1-\lambda/\left(1+p\right)}{1+\min\limits_{1\le j\le n}\left[B_j\right]}\right]$$

Consequently, we have assertion (6.7.1) with the theorem (6.2.1).For $A_j = A$ and $B_j = B$,where $j = 1,2,3,....m$ .The theorem yields.

COROLLARY 6.7.1: Let the functions $f_j\left(z\right)$ where $j = 1,2,3,....m$ is defined by $f_j\left(z\right) = z^p - \sum_{n=1}^{\infty}\left|a_{p+n,j}\right|z^{p+n}$, be in the class $J\left(A,B,p,\delta\right)$ .Also let

$$\left\{\frac{\delta}{1+R}+B\right\} \ge 0$$

Then

$$\left(f_1^*,.....f_m\right)\left(z\right) \in j\left(A^m, B^m, p, \delta\right)$$

THEOREM 6.7.2 : Let the function f and g defined by (6.1.1) and (6.1.2) respectively be in the class $J\left(A,B,p,\delta\right)$. Then $f*g\left(z\right)$ is defined by $\left(6.1.3\right)$ belong to the class $J\left(A,B,p,\delta\right)$.with $-1 \le A < B \le 1$ where

$$A \le A_1\left(p,A,B,\delta\right) = B + \frac{\left(B-A\right)^2}{\left(1+B\right)}\frac{\left(1+B-\delta\right)}{\left(1+p\right)}$$

The result is sharp.

PROOF: Since $f.g \in J(A,B,p,\delta)$ .Then the theorem5.2.1,we have

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)\frac{(1+B)}{(B-A)}|a_{p+n}| \le 1$$

(6.7.4)

And

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)\frac{(1+B)}{(B-A)}|b_{p+n}| \le 1$$

(6.7.5)

We have wish to find the value A_1, such that $1- \le A_1 < B$ for $(f*g) \in J(A_1,B,p,\delta)$. Equivalently, we want to determine $A_{1,}, B$ satisfying

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)\frac{(1+B)}{(B-A)}|b_{p+n}| \le 1$$

(6.7.6)

Combining the (6.7.4) and (6.7.5) ,we get the inequality

$$\sum_{n=1}^{\infty}u\sqrt{|a_{p+n}||b_{p+n}|}$$

(6.7.7)

Where

$$\le \left\{\sum_{n=1}^{\infty}u|a_{p+n}|\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}u|b_{p+n}|\right\}^{\frac{1}{2}} \le 1$$

$$u = \phi(n,p,\delta)\frac{(1+B)}{(B-A)}$$

(6.7.6) is satisfied if

$$u_1|a_{p+n}||b_{p+n}| \le u\sqrt{|a_{p+n}||b_{p+n}|}$$

$$u_1 = \phi(n,p,\delta)\frac{(1+B)}{(B-A_j)}$$ for n∈N.

But from (6.7.7), we have

$$\sqrt{|a_{p+n}||b_{p+n}|} \le \frac{1}{u},(n \in N)$$

Therefore, it is enough to find u_1 such that

$$\frac{1}{u} \le \frac{u}{u_1}$$

or

$u_1 \le u^2$

or it is equivalent to

$$B - A \le \frac{(B-A)^2 \Gamma(1+p)\Gamma(n+1+p-\delta)}{(1+B)\Gamma(n+1+P)\Gamma(1+p-\delta)} \qquad \text{for } n \ge 1$$

(6.7.8)

Right hand member decreases as n increases and so it is maximum for $n = 1$,so (6.7.8) is satisfied and proved

$$B - A \ge \frac{(1+p-\delta)}{(1+p)} \frac{(B-A)^2}{(1+B)}$$

Or

$$A_1 \le B + \frac{(B-A)^2}{(1+B)} \frac{(1+p-\delta)}{(1+p)}$$

This proves the desired result. The result is sharp for the function

$$f(z) = g(z) = z^p - \frac{(B-A)}{(1+B)} \frac{(1+p-\delta)}{(1+p)} z^{p-1}$$

CORROLARY6.7.2 : Let $f, g \in J(A, B, p, 1)$ .Then $(f * g)(z) \in J(A, B, p, 1)$.

Where $A_1 \le A_1(p, A, B) = B + \frac{(B-A)^2 p}{(1+B)(1+p)}$

The result is sharp for the function.

$$f(z) = g(z) = z^p - \frac{(B-A)}{(1+B)} \frac{(p)}{(1+p)} z^{p+1}$$

Let $f, g \in J(A, B, p, o)$ .Then $(f * g)(z) \in J(A, B, p, o)$.

Where

$$A_1 \le A_1(A, B) = B + \frac{(B-A)}{(1+B)} z^{p-1}$$

THEOREM6.7.3: If $f \in J(A, B, p, \delta)$ and $g \in J(A, B, p, \delta)$. Then

$(f * g)(z) \in J(A_2, B, p, \delta)$.

$(f * g)(z) \in J \in (A_2, B, p, \delta)$

Where

$$A_2 \le A_2\left(p,A,B,\delta\right) = B + \frac{\left(B-A\right)^2\left(1+p-\delta\right)}{\left(1+B\right)\left(1+p\right)}$$

The result is possible for

$$f(z) = z^p - \frac{\left(B-A\right)\left(1+p-\delta\right)}{\left(1+B\right)\left(1+p\right)} z^{p-1}$$

and

$$g(z) = z^p - \frac{\left(B-A'\right)\left(1+p-\delta\right)}{\left(1+B\right)\left(1+p\right)} z^{p-1}$$

PROOF: Proceeding exactly as in theorem (6.7.2),

We require

$$\frac{\Gamma\left(n+1+p\right)\Gamma\left(1+p-\delta\right)}{\Gamma\left(1+p\right)\Gamma\left(n+1+p-\delta\right)}\frac{\left(1+B\right)}{\left(B-A_2\right)}$$

$$\le \frac{\left(1+B\right)}{\left(B-A\right)}\frac{\Gamma\left(n+1+p\right)\Gamma\left(1+p-\delta\right)}{\Gamma\left(1+p\right)\Gamma\left(n+1+p-\delta\right)}\frac{\Gamma\left(n+1+p\right)}{\Gamma\left(n+1+p-\delta\right)}\left\{\frac{\left(1+B\right)}{\left(B-A\right)}\right\}.n \ge 1$$

That

$$B - A_2 \ge \frac{\left(B-A\right)^2}{\left(1+B\right)}\frac{\Gamma\left(1+p\right)\Gamma\left(n+1+p-\delta\right)}{\Gamma\left(n+1+p\right)\Gamma\left(1+p-\delta\right)}$$

(6.7.9)

The right hand side member decreases as n increases and so it is maximum for n=1.So the (6.7.9) it is satisfied proved of the given relations.

$$B - A_2 \ge \frac{\left(1+p-\delta\right)}{\left(1+p\right)}\frac{\left(B-A\right)\left(B-A'\right)}{\left(1+B\right)}$$

or

$$A_2 \le B + \frac{\left(1+p-\delta\right)}{\left(1+p\right)}\frac{\left(B-A\right)\left(B-A'\right)}{\left(1+B\right)}.$$

COROLLARY 6.7.3: If $f,g,n \in J\left(A,B,p,\delta\right)$.

Then

$\left(f*g*n\right)\left(z\right) \in J\left(A_3,B,p,\delta\right)$

, where

$$A_3 \le A_3\left(p,A,B,\delta\right) = B + \frac{\left(B-A\right)^3}{\left(1+B\right)}\frac{\left(1+p-\delta\right)}{\left(1+p\right)^2}$$

The result is best possible for

$$f(z)=g(z)=n(z)=z^{p}-\frac{(B-A)}{[1+B]}\frac{(1+p-\delta)}{(1+p)}z^{p+1}$$

(6.7.10)

THEOREM 6.7.4 : Let the function f defined by (6.1.1) be in the class $J(A,B,p,\delta)$.

We also let

$$g(z)=z^{p}-\sum_{n=1}^{\infty}|b_{p+n}|z^{p+1}.$$

Where$\left(|b_{p+n}|\leq 1\right); p\in N$

Then$(f*g)(z)$ belongs to the class $J(A,B,p,\delta)$.

PROOF: Since

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)|a_{P+n}||b_{p+n}|$$

$$\leq\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)|a_{P+n}|\leq(B-A).$$

Hence

$$(f*g)(z)\in J(A,B,p,\delta)$$

THEOREM 6.7.5: Let the function f and g defined by(6.1.1) and(6.1.2) respectively belongs to $J(A,B,p,\delta)$.Then

$$h(z)=z^{p}-\sum_{n=1}^{\infty}\left[|a_{p+n}|^{2}+|b_{p+n}|^{2}\right]z^{p+n}(p\in N)$$

Belongs to the class $J(A_4,B,p,\delta)$.

COROLLARY 6.7.5 : Let $f\in J(A,B,p,\delta)$ .Also

$$g(z)=z^{p}-\sum_{n=1}^{\infty}|b_{p+n}|z^{p+n},\left(0\leq|b_{p+n}|z^{p+n}\right)$$

Then$(f*g)(z)$belongs to class $J(A,B,p,\delta)$.

$$A_4\leq A_4(p,A,B,\delta)=B+\frac{2(B-A)^{2}(1+p-\delta)}{(1+B)(1+p)}$$

The result is sharp for the function.

$$f(z)=g(z)=z^{p}-\frac{(B-A)}{(1+B)}\frac{(1+p-\delta)}{(1+p)}z^{p+1}$$

PROOF: In view of theorem(6.7.2),it is sufficient to prove that

$$\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+p-\delta)}\frac{(1+B)}{(B-A_4)}\left[\left|a_{p+n}\right|^2+\left|b_{p+n}\right|^2\right]\leq 1$$

(6.7.11)

Where A_4 is defined by(6.7.10)

Since $f,g\in J(A,B,p,\delta)$ .Then

$$\sum_{n=1}^{\infty}\left[\phi(n,p,\delta)\frac{(1+B)}{(B-A)}\right]^2\left|a_{p+n}\right|^2\leq 1$$

and

$$\sum_{n=1}^{\infty}\left[\phi(n,p,\delta)\frac{(1+B)}{(B-A)}\right]\left|b_{p+n}\right|^2\leq 1$$

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\phi(n,p,\delta)\frac{(1+B)}{(B-A)}\right]\left\{\left|b_{p+n}\right|^2+\left|b_{p+n}\right|^2\right\}\leq 1 \qquad (6.7.12)$$

By comparing (6.7.11) and (6.7.12),it is easily seen that inequality (6.7.11) will be satisfied if

$$\phi(n,p,\delta)\frac{(1+B)}{(B-A_4)}\leq\left\{\phi(n,p,\delta)\right\}^2\left\{\frac{(1+B)}{(B-A)}\right\}^2,n\in N$$

That is if

$$(B-A_4)\geq\frac{2(B-A)^2}{(1+B)}\frac{\Gamma(1+p)\Gamma(n+1+p-\delta)}{\Gamma(n+1+p)\Gamma(1+p-\delta)},$$

Then right hand member decrease as n increase and so it is maximum for n=1.So the above inequality is satisfied if,

$$A_4\leq B+\frac{2(B-A)^2}{(1+B)}\frac{(1+p-\delta)}{(1+p)}$$

This is complete the proof of the theorem.

6.8 CONTENTMENT RELATION:

With the help of the theorem (6.2.1).We immediately obtain the following theorems.

THEOREM 6.8.1 : $0 \le \delta \le 1, -1 \le A < 1 and -1 \le B < 1$.Then

$$J(A, B_1, p, \delta) = J\left(\frac{1 - B + 2A}{1 + B}1.p.\delta\right)$$

Generally we can see that if $-1 \le A < 1 and 0 \le B < 1$. Then

$$J(A, B, p, \delta) = J(A; B; p, \delta)$$

If and only if

$$\frac{(B - A)}{(1 + B)} = \frac{(B' - A')}{(1 + B')}$$

COROLLARY 6.8.1: Let $0 \le \delta \le 1, -1 \le A_1 \le A_2 < 1$ and $0 \le B \le 1$. Then $J(A_1, B, p, \delta) \supset J(A_2, B, p, \delta)$.

COROLLARY6.8.2: $0 \le \delta \le 1, -1 \le B_1 \le B_2 < 1$. Then

$$J(A, B_1, p, \delta) \supset J(A_2, B_2, p, \delta).$$

6.9 CLOSURE THEOREM:

THEOREM6.9.1: Let $f_j(z) = z - \sum_{n=1}^{\infty} |a_{p+n,j}| z^{p+n}$, $(j = 1, 2, 3, ...m; p \in N)$

If $F_j \in J(A, B, p, \delta)$,for each $(j = 1, 2, 3, ...m; p \in N)$.Then the function

are also belong to $J(A, B, p, \delta)$.

Where,

$$|b_{p+n}| = \frac{1}{m}\sum_{j=1}^{\infty} |a_{p+n,j}| .$$

PROOF: Since $F_j \in J(A, B, p, \delta)$.It follows from the theorem(6.2.1).

We have

$$\sum_{n=1}^{\infty} \phi(n, p, \delta)(1 + B)|a_{p+n,j}| \le (B - A) \text{ for each } (j = 1, 2, 3, ...m; p \in N)$$

Therefore

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)|b_{p+n}|$$

$$\leq\sum_{n=1}^{\infty}\delta(n,p,\delta)(1+B)\left\{\frac{1}{m}\sum_{j=1}^{\infty}|a_{p+n,j}|\right\}$$

$<(B-A)$ by, theorem(6.2.1)

Hence $n\in J(A,B,p,\delta)$.

THEOREM 6.9.2: The class $J(A,B,p,\delta)$ is convex.

PROOF: Let f and $g\in J(A,B,p,\delta)$.The it is sufficient to show that the function.

$$n(z)=z^{p}-\sum_{n=1}^{\infty}\left\{u|a_{p+n}|+(1-u)|b_{p+n}|\right\}z^{p+n}.$$

And $(0\leq u\leq 1)$ is also the class $J(A,B,p,\delta)$.Since f and $g\in J(A,B,p,\delta)$ .Then from the theorem (6.2.1).We have

$$\sum_{n=1}^{\infty}\phi(n.p,\delta)(1+B)|b_{p+n}|\leq(B-A).$$

Therefore

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)\left\{u|a_{p+n}|+b(1-u)|b_{p+n}|\right\}.$$

$\leq(B-A)$

Hence $n\in J(A,B,p,\delta)$.

COROLLARY 6.9.2: The extreme point of the class $J(A,B,p,\delta)$ is the function of the form

$$f_{p+n}(z)=z^{n}-\frac{(B-A)}{(1+B)\phi(n,p,\delta)}z^{p+n},n\in N(0)$$

THEOREM 6.9.3: Let $f_{p}(z)=z^{p}$ and

$$f_{p+n}(z)=z^{p}-\frac{(B-A)}{(1+B)\phi(n,p,\delta)}z^{p+n}\text{ Where }-1\leq A<B\leq 0\leq\delta\leq 1;p\in N.$$

Then the function in the class $J(A,B,p,\delta)$ if and only if, it can be expressed of the form

$$f(z)=\sum_{n=0}^{\infty}c_{p+n}f_{p+n}(z).\text{Where }c_{p+n}\geq 0;\sum_{n=0}^{\infty}c_{p+n}=1.$$

PROOF: Let us suppose that

$$f(z)=\sum_{n=0}^{\infty}c_{p+n}f_{p+n}(z)$$

$$=z^{p}-\sum_{n=1}^{\infty}\frac{(B-A)}{(1+B)\phi(n,p,\delta)}c_{p+n}z^{p+n}$$

Then

$$\sum_{n=1}^{\infty}\phi(n,p,\delta)(1+B)\left\{\frac{(B-A)c_{p+n}}{(1+B)\phi(n,p,\delta)}\right\}$$

Hence, by theorem(6.2.1), $f\in J(A,B,p,\delta)$ conversely. Let $f\in j(A,B,p,\delta)$,

It follows then from theorem6.2.1 that

$$|a_{p+n}|\leq\frac{(B-A)}{\phi(n,p,\delta)(1+B)},n\in N$$

Then, we have

$$c_{p+n}=\frac{(1+B)\phi(n,p,\delta)}{(B-A)},n\in N$$

And

$$c_{p}=1-\sum_{n=1}^{\infty}c_{p+n}$$

Now, we have

$$f(z)=\sum_{n=0}^{\infty}c_{p+n}f_{p+n}(z)$$

This is complete proof of the theorem.

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CHAPTER 7

ANALYTIC FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVE (II)

7.1 INTRODUCTION:

In this chapter, we have also introduced a new class $\mathrm{H}(A,B,p,\delta)$ of Analytic Function defined Fractional Derivative, as defined below. A function $f(p)$ belongs to the class $H(A,B,p,\delta)$.If and only if there exists a function w belonging to the class $H(A,B,p,\delta)$ such that

$$\frac{\Omega_Z^{(\delta,p)} f(z)}{\Omega_Z^{(\delta-1,p)} f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, z \in u$$

(7.1.1)

Where $-1 < A < B < 1$ and $\Omega_Z^{(\delta,p)} f(z)$ is defined by (5.1.5)

The condition (7.1.1) is equivalent to

$$\left| \frac{\Omega_Z^{(\delta,p)} f(z) - \Omega_Z^{(\delta-1,p)} f(z)}{B\Omega_Z^{(\delta,p)} f(z) - A\Omega_Z^{(\delta-1,p)} f(z)} \right| < 1, z \in u$$

(7.1.2)

By giving the specific values to A, B, p and δ in (7.1.2), we obtain the following important sub class studied by various researchers in earlier works.

(1) For $\delta = 1$,we obtain the class of functions $f(z)$ is satisfying the conditions

$$\left| \frac{zf'(z) - f(z)}{Bf'(z) - Af'(z)} \right| < 1, z \in u$$

Studied by Goel, and Sohi, For $\delta = 1, A = (2\alpha - 1)\beta, B = \beta$ and $p = 1$,we obtain the class of function $f(z)$ is satisfying the conditions

$$\left| \frac{zf'(z) - f(z)}{zf'(z) - (2\alpha - 1) f(z)} \right| < \beta, z \in u$$

Where $0 < \alpha < 1$ and $0 < \beta < 1$ is studied by Gupta V.P and Jain P.K[87]

(2) For $\delta = 1, A = (2\alpha - 1)\beta, B = 1$, we obtain the class of function $f(z)$ is satisfying the conditions.

$$\left|\frac{zf'(z) - f(z)}{zf'(z) - (2\alpha - 1)f(z)}\right| < 1, z \in u$$

Studied by Silverman, H . This chapter is divided into nine sections for the systematic study of the class $H(A,B,p,\delta)$. In sections 7.2,we have obtained the necessary and sufficient conditions in terms of coefficients for a functions $f(z)$ is belonging to the $H(A,B,p,\delta)$.In sections 7.3,we have investigated the Distortion Properties for the class $H(A,B,p,\delta)$.In sections7.4,we study the Integral Operator of the form (6.1.8).In Sections7.5, we have investigated the radius of p-valent Star likeness for the class $H(A,B,p,\delta)$.In sections 7.6,we have determined the p-valent convexity for the class $H(A,B,p,\delta)$.In sections7.7,we have obtained the result involving the Modified Hadamard Product of two functions belonging to the class $H(A,B,p,\delta)$.In sections7.8, we have obtained the class some contentment relations related to the class $H(A,B,p,\delta)$. In sections7.9, we have investigated the Closure Properties for the class $H(A,B,p,\delta)$

Note : Throughout this chapter, we assume that

$$\psi(n,p,\delta) = \frac{\Gamma(n+1+p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+2+p-\delta)}$$

7.2 NECESSARY AND SUFFICIENT CONDITION

THEOREM 7.2.1 A functions f(z) is defined by(7.2.1)in the class $H(A,B,p,\delta)$ if and only if

$$\sum_{n=1}^{\infty} \psi(n,p,\delta)\{(1+B)n + (B-A)(1+p-\delta)\}|a_{p+n}| \le (B-A) \quad (7.2.1)$$

PROOF: Let $|z| = 1$. Then

$$\left|\Omega^{(\delta,p)} f(z) - \Omega_z^{\delta-1,p}\right| - \left|B\Omega_z^{\delta,p} f(z) - A\Omega_z^{\delta-1,p} f(z)\right|$$

$$= \left|-\sum_{n=1}^{\infty} \psi(n,p,\delta) n |a_{p+n}| z^n\right| - \left|(B-A) - \sum_{n=1}^{\infty} (n,p,\delta)\{Bn + (B-A)(1+p-\delta)\}|a_{p+n}| z^n\right|$$

$$\leq \sum_{n=1}^{\infty} \psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}\left|a_{p+n}\right|-(B-A)$$

$$< 0$$

Hence, by the Maximum Modulus Theorem,

$f \in H(A,B,p,\delta)$.

To prove the converse, let

$$\left|\frac{\Omega_z^{(\delta,p)} f(z)-f(z)\Omega_z^{(\delta-1,p)} f(z)}{B\Omega_z^{(\delta,p)} f(z)-A\Omega_z^{(\delta-1,p)} f(z)}\right|$$

$$\left|\frac{-\sum_{n=1}^{\infty} \psi(n,p,\delta)n\left|a_{p+n}\right|z^n}{(B-A)-\sum_{n=1}^{\infty} \psi(n,p,\delta)\{Bn+(B-A)(1+p-\delta)\}\left|a_{p+n}\right|z^n}\right|$$

Since $\left|\operatorname{Re}(z)\right| < |z|$ for all z, we have

$$\operatorname{Re}\left\{\frac{\sum_{n=1}^{\infty} \psi(n,p,\delta)n\left|a_{p+n}\right|z^n}{(B-A)-\sum_{n=1}^{\infty} \psi(n,p,\delta)\{Bn+(B-A)(1+p-\delta)\}\{a_{p+n}\}z^n}\right\}<1 \tag{7.2.2}$$

Choose the values of z on the real axis, so

that $\left|\frac{\Omega_z^{(\delta,p)} f(z)}{\Omega_z^{(\delta-1,p)} f(z)}\right|$ is real. Once clearing the denominator of the (7.2.2) and letting z=1 through real values, we obtain

$$\sum_{n=1}^{\infty} \psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\} \leq (B-A)$$

This is the complete proof of the theorem.

COROLLARY7.2.1: Let the function $f(z)$ defined by (7.1.1) be in the class $H(A,B,p,\delta)$.

Then

$$\left|a_{p+n}\right| \leq \frac{(B-A)}{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)} \text{ for, } n \in N$$

7.3 DISTORTION THEOREM:

THEOREM 7.3.1: Let the function $f(z)$ defined by (7.1.1) be in the class $H(A,B,p,\delta)$.

Then

$$\left|z^p\right|-\frac{(B-A)(2+p-\delta)(1+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}(1+p)}|z|^{p+1}$$

(7.3.1)

$$\le |f(z)| \le |z|^p-\frac{(B-A)(2+p-\delta)(1+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}(1+p)}|z|^{p+1}$$

And

$$\frac{\Gamma(1+p)}{\Gamma(1+p-\delta)}|z|^{p-\delta}\left[1-\frac{(B-A)(2+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}}\right]|z|^{p+1}$$

$$\le\left|D_z^{\delta}f(z)\right|\le\frac{\Gamma(1+p)}{\Gamma(1+p-\delta)}|z|^{p-\delta}\left[1-\frac{(B-A)(2+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}}\right]|z|^{p+1}$$

(7.3.2)

Whenever $z\in u$

PROOF: Since f(z) belongs to the class $H(A,B,p,\delta)$. In view of Theorem (7.2.1).we have

$$\{(1+B)+(B-A)(1+p-\delta)\}\psi(1,p,\delta)\sum_{n=1}^{\infty}\left|a_{p+n}\right|$$

(7.3.3)

$$\le\sum_{n=1}^{\infty}\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)\left|a_{p+n}\right|\le(B-A)$$

This yield

$$\sum_{n=1}^{\infty}\left|a_{p+n}\right|\le\frac{(B-A)}{\{(1+B)n+(B-A)(1+p-\delta)\psi(1,p,\delta)\}}$$

Hence

$$|f(z)|\ge|z|^p-\sum_{n=1}^{\infty}\left|a_{p+n}\right||z|^{p+n}$$

$$\geq |z|^p - |z|^{p+1}\sum_{n=1}^{\infty}|a_{p+n}|$$

$$\geq |z|^p - \frac{(B-A)}{\{(1+B)+(B-A)(1+p-\delta)\psi(1,p,\delta)\}}|z|^{p+1}$$

And

$$f(z) \geq |z|^p + \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$$

$$\geq |z|^p + \frac{(B-A)}{\{(1+B)+(B-A)(1+p-\delta)\psi\{1,p,\delta\}\}}|z|^{p+1}.$$

This gives the inequality (7.3.1).

Next, by using the second inequality in(7.3.3),we observe the that

$$\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+1+p-\delta)}|a_{p+n}| \leq \frac{(B-A)(2+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}}.$$

Now

$$\left|z^p\Omega_z^{(\delta,p)}f(z)\right| \geq |z|^p - \sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+1+p-\delta)}|a_{p+n}||z|^{p+n}$$

$$\geq |z|^p - |z|^{p+1}\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+1+p-\delta)}|a_{p+n}|$$

$$\geq |z|^p - \frac{(B-A)(2+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}}|z|^{p+1}$$

And

$$\left|z^p\Omega_z^{(\delta,p)}f(z)\right|' \geq |z|^p + \sum_{n=1}^{\infty}\frac{\Gamma(n+1-p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+1+p-\delta)}|a_{p+n}||z|^{p+n}$$

$$\geq |z|^p + |z|^{p+n}\sum_{n=1}^{\infty}\frac{\Gamma(n+1-p)\Gamma(1+p-\delta)}{\Gamma(1+p)\Gamma(n+1+p-\delta)}|a_{p+n}|$$

$$\geq |z|^p + \frac{(B-A)(2+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}}|z|^{p+n}.$$

This gives the inequality (7.3.2).

COROLLARY 7.3.1 Under the hypothesis of the Theorem (7.3.1), the function $f(z)$ is included in a disc with the centre at the origin and radius r is given by

$$r = 1 + \frac{(B-A)(2+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}}$$

And it is included in a disc with its centre at the origin and the radius R is given by

$$R = \frac{\Gamma(1+p)}{\Gamma(1+p-\delta)}\left[1 + \frac{(B-A)(2+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}}\right]$$

7.4 INTEGRAL OPERATOR:

THEOREM 7.4.1 Let the function $f(z)$ defined by (7.1.1) is in the class $H(A,B,p,\delta)$.Also let c>-p. Then the function F defined by (7.1.8) is also in the class $H(A,B,p,\delta)$.

PROOF: From the definition (7.1.8) and (7.1.1),it is easily seen that

$$F(z) = z^p - \sum_{n=1}^{\infty} |h_{p+n}| z^{p+n},$$

Where

$$|h_{p+n}| = \frac{(c+p)}{(c+p+n)}|a_{p+n}|$$

Therefore

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \psi(n,p,\delta)\{(1+B)n + (B-A)(1+B-\delta)\}|h_{p+n}| \\
&= \sum_{n=1}^{\infty} \psi(n,p,\delta)\{(1+B)n + (B-A)(1+p-\delta)\}\frac{(c+p)}{(c+p+n)}|a_{p+n}| \\
&< \sum_{n=1}^{\infty} \psi(n,p,\delta)\{(1+B)n + (B-A)(1+p-\delta)\}|a_{p+n}| \\
&< (B-A), by Theorem (7.2.1)
\end{aligned}$$

Hence $F \in H(A,B,p,\delta)$.

Theorem (7.4.1) simplifies considerably when we have the set $c=1-p$ and thus we obtain

COROLLARY 7.4.1 : If $F \in H(A,B,p,\delta)$. Then

$$F(z)=z^{p-1}\int_0^{\infty}\left[\frac{f(t)}{t^p}\right]dt \in H(A,B,p,\delta).$$

In the following theorem, we consider the converse problem of the above theorem.

THEOREM 7.4.2 Let $C>-p$. Also let function $f(z)$ is in the class $H(A,B,p,\delta)$. Then the $F(z)$ given by (6.1.8) is p-valent in the disc $|z|<R$, where

$$R=\inf_{n\in N}\left[\frac{\{(1+B)n+(B-A)(1+p-\delta)(c+p)p\psi(n,p,\delta)\}}{(B-A)(c+p+n)(p+n)}\right]$$

The result is sharp.

PROOF: Let $F(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$ belongs to the $H(A,B,p,\delta)$. Then the theorem (7.1.8), it follows that

$$f(z)=\frac{z^{1-c}\{z^c F(z)\}'}{(c+p)}=|z|^p-\sum_{n=1}^{\infty}\left\{\frac{(c+p+n)}{(c+p)}\right\}|a_{p+n}|z^{p+n}$$

In order to established the required result, it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p \quad for|z|<R$$

Now

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le\sum_{n=1}^{\infty}\frac{(p+n)(c+p+n)}{(c+p)}|a_{p+n}|z^n$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p, if$$

$$\sum_{n=1}^{\infty}\frac{(p+n)(c+p+n)}{(c+p)}|a_{p+n}|z^p<p$$

But from the theorem (7.2.1), we have

$$\sum_{n=1}^{\infty}\psi(n,p,\delta)\frac{\{(1+B)n+(B-A)(1+p-\delta)\}}{(B-A)}|a_{p+n}|\leq p.$$

This inequality (7.4.1) will be satisfied if

$$\frac{(p+n)(c+p+n)}{(c+p)}|a_{p+n}|z^n$$
$$<\psi(n,p,\delta)\frac{\{(1+B)n+(B-A)(1+p-\delta)\}p}{(B-A)}$$

For each $n\in N, or$ if

$$|z|<\left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}(c+p)p\psi(n,p,\delta)}{(B-A)(p+n)(c+p+n)}\right]^{\frac{1}{n}}$$

For each $n\in N$.

Hence $f(z)$ is $p-valent$ for each $|z|\in R$. To show the sharpness of the result, we take,

$$F(z)=z^p-\frac{(B-A)}{\{(1+B)n+(B-A)(1+p-\delta)\psi(n,p,\delta)\}}$$

For each $n\in N$.

Clearly $F\in H(A,B,p,\delta)$ and thus

$$f(z)=z^p-\frac{(c+p+n)(B-A)z^{p+n}}{(c+p)(1+B)n+(B-A)(1+p-\delta)\psi(n,p,\delta)},n\in N.$$

Therefore

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|\frac{(B-A)(c+p+n)(p+n)z^n}{\{(1+B)n+(B-A)(1+p-\delta)\}(c+p)\psi(n,p,\delta)}\right|$$

=p at z=R.

Hence the result is sharp.

7.5 RADIUS OF P-VALENT STARLIKENESS:

THEOREM 7.5.1: Let the function $f(z)$ defined by (7.1.1) be in the class $H\left(A,B,p,\delta\right)$ .Then the function $f(z)$ is p-valent starlike in the disc $|z|<R^*$,where

$$R^* = \inf_{n\in N}\left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)(n+p)}\right]^{\frac{1}{n}}$$

The result is sharp. $\geq |z|^p + \dfrac{(B-A)(2+p-\delta)}{\{(1+B)+(B-A)(1+p-\delta)\}}|z|^{p+n}$.

PROOF: In order to obtain the required result, it is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-p\right|< \text{ for } |z|<R^*$$

Let $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$.

Then we have

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq\frac{\sum_{n=1}^{\infty}n|a_{p+n}||z|^n}{1-\sum_{n-1}^{\infty}|a_{p+n}||z|^n}$$

Therefore

$$\left|\frac{zf'(z)}{f(z)}-p\right|<p \quad \text{if}$$

$$\sum_{n=1}^{\infty}\frac{(n+p)}{(p)}|a_{p+n}||z|^n<1$$

(7.5.1)

But from Theorem (7.2.1), we have

Hence (7.5.1) will be satisfied if

$$\frac{(n+p)}{p}|a_{p+n}||z|^n < \psi(n,p,\delta)\frac{\{(1+B)n+(B-A)(1+p-\delta)\}}{(B-A)}, \text{ for each } n\in N$$

,or if

$$|z| < \left|\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)(n+p)}\right|^{\frac{1}{n}}, \text{ for each } n\in N$$

Then

$$\left|\frac{zf'(z)}{f(z)}-p\right| = \left|\frac{\dfrac{-n(B-A)z^n}{\{(1+B)n+(B-A)(1+p-\delta)\psi(n,p,\delta)\}}}{1-\dfrac{(B-A)z^n}{\{(1+B)n+(B-A)(1+p-\delta)\psi(n,p,\delta)\}}}\right|$$

$= p$. At $z = R^*$

Hence the result is sharp.

7.6 RADIUS OF p-VALENT CONVEXITY:

THEOREM7.6.1: Let the function $f(z)$ defined by (5.1.1)be in the class $H(A,B,p,\delta)$. Then the function $f(z)$ is p-valent convex in the disc $|z| < R^*$

,where $R_1^* Inf_{n\in N}\left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)p^2}{(B-A)(p+n)^2}\right]^{\frac{1}{n}}$

The result is sharp.

PROOF: In order to established the required result, it is sufficient to show that

$$\left|1+\frac{zf'(z)}{f'(z)}\right| < p \text{ for } |z| < R_1^*$$

Let $f(z) = z^n - \sum_{n=1}^{\infty}|a_{p+n}|z^n$, then we have

$$\left|1+\frac{zf'(z)}{f'(z)}-p\right| \le \frac{\sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^n}{p-\sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^n}$$

Therefore

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| < p, if$$

$$\sum_{n=1}^{\infty}\frac{(p+n)^2}{p^2}|a_{p+n}||z|^n < 1$$

(7.6.1)

But from the Theorem (7.2.1), we have

$$\sum_{n=1}^{\infty}\psi(n,p,\delta)\frac{\{(1+B)n+(B-A)(1+p-\delta)\}}{(B-A)}|a_{p+n}| \le 1$$

Hence (7.6.1) will be satisfied if

$$\frac{(p+n)^2}{p^2}|a_{p+n}||z|^n < \frac{\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}}{(B-A)}|a_{p+n}|$$

For each $n \in N$

$$|z| < \left[\frac{\{(1+B)\}n+(B-A)(1+p-\delta)\psi(n,p,\delta)}{(B-A)(p+n)^2}\right]^{\frac{1}{n}}$$

For each $n \in N$

Therefore $f(z)$ is convex in $|z| < R_1^*$.

To show that sharpness of the result, we take

$$F(z) = z^p - \frac{(B-A)}{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}z^{p+n}, n \in N$$

Then

$$\left|1+\frac{zf''(z)}{f(z)}-p\right| = \left|\frac{\dfrac{-n(p+n)(B-A)z^n}{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}}{p-\dfrac{(p+n)(B-A)z^n}{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}}\right|$$

$p = 0$ $z = R^*$, which show that the result is sharp.

7.7 SOME RESULTS INVOLVING MODIFIED HADAMARD PRODUCT:

In the following theorems, we use the technique of Padmanabhann.

THEOREM7.7.1: Let the functions f and g defined by (7.1.1) and(7.1.2) respectively be in the class $H(A,B,p,\delta)$. Then $f*g$ defined by (7.1.3) belongs to class $H(A,B,p,\delta)$,where

$$A_1 \le 1-2k \text{ and } B_1 \ge \frac{A_1+k}{1-k}$$

With

$$k = \frac{(1+p-\delta)^2(B-A)^2}{\left[(p+1)^2(1+B)^2(1+p-\delta)(B-A)(2+B+A)+\delta(1+p-\delta)(2+p-\delta)(B-A)^2\right]}$$

PROOF : Since $f,g \in H(A,B,p,\delta)$. Then from theorem (7.2.1), we have

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)(1+B-\delta)\}\psi(n,p,\delta)}{(B-A)}|a_{p+n}| \le 1 \qquad (7.7.1)$$

and

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}|b_{n+1}| \le 1 \qquad (7.7.2)$$

Our aim is to find that the valuesA_1,B_1 such that $-\le A_1 < B_1 \le 1, for(f*g)H(A_1,B_1,p,\delta)$. Equivalently, we want to determine A_1, B_1 satisfying

$$\sum_{n=1}^{\infty} \frac{\{(1+B_1)n+(B_1-A_1)(1+p-\delta)\}\psi(n,p,\delta)}{(B_1-A_1)}|a_{p+n}||b_{p+n}| \le 1.$$

Combining (77.1) and (7.7.2), we get using the Cauchy-Schwarz inequality

$$\sum_{n=1}^{\infty} u\sqrt{|a_{p+n}||b_{p+n}|} \le \left\{\sum_{n=1}^{\infty} u|a_{p+n}|\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\infty} u|b_{p+n}|\right\}^{\frac{1}{2}} \le 1$$

where

$$u = \frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}$$

(7.7.3)

For each $n \in N$,(7.7.3) is satisfied if

$$u_1 = \frac{\{(1+B_1)n+(B_1-A_1)(1+p-\delta)\}\psi(n,p,\delta)}{(B_1-A_1)}, n \in N.$$

But from (7.7.4) ,we have

$$\sqrt{|a_{p+n}||b_{p+n}|} \le \frac{1}{u}, n \in u$$

Therefore, it is enough to find u_1 such that

$$\frac{1}{u} \le \frac{u}{u_1}$$

or

$$u_1 < u^2$$

It is equivalent to

$$\frac{\{(1+B_1)n+(B_1-A_1)(1+p-\delta)\}}{(B_1-A_1)}\psi(n,p,\delta) \le \left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}\right]^2$$

$$u^2, n \ge 1$$

(7.7.4)

That is

$$\{(1+B_1)n+(B_1-A_1)(1+p-\delta)\}\psi(n,p,\delta) \le u^2(B_1-A_1)$$

This yield

$$A_1 < \frac{u^2B_1-(1+p-\delta-n)\psi(n,p,\delta)B_1-n\psi(n,p,\delta)}{u^2-(1+p-\delta)\psi(n,p,\delta)}$$

It is easily to varify that

$$u^2 > (1+p-\delta)\psi(n,p,\delta) \text{ for } n \ge 1$$

Now the above inequality gives the simplification.

$$\frac{(B_1 - A_1)}{(1+B_1)} \geq \frac{n\psi(n,p,\delta)}{u^2 - (1+p-\delta)\psi(n,p,\delta)} \text{ for } n \geq 1 \qquad (7.7.5)$$

The right hand member decrease as n increases and so is maximum for n=1 .So (7.7.5) is satisfied.

$$\frac{B_1 - A_1}{1+B_1} \geq \frac{(1+p-\delta)^2 (B-A)^2}{\left[(1+p)(B-A)^2 + (p+1)(1+p-\delta)(B-A)(2+B+A) + \delta(1+p-\delta)(2+p-\delta)(B-A)\right]^2} \qquad (7.7.6)$$

$$= k$$

Obviously k<1 fixing A_1 in (7.7.6), we get.

Let $B_1 = 1,$ then $A_1 \leq 1 - 2k.$

Therefore $(f * g)(z) \in H(1-2k, 1, p, \delta),$ with k defined as (7.7.6).

COROLLARY7.7.1: Let the functions f and g defined by (7.1.1) and (7.1.2) respectively, be in the class $H(A,B,p,1)$. Then $(f*g)(z)$ defined by (7.1.3) belongs to the class $H(1-2k,1,p,\delta)$.where

$$k = \frac{p(B-A)^2}{(1+B)^2 + 2p(B-A)(1+B-A)}$$

THEOREM7.7.2 : Let function f defined(7.1.1) belongs to the class $H(A,B,p,\delta)$ and g defined by (7.1.2) belongs to the class $H(A',B',p,\delta)$,then $(f*g)(z)$ defined by (7.1.3)belongs to the class $H(A_2,B_2,p,\delta)$ .where $A_2 \leq 1-2k$ and B_2 ,and $B_2 \geq \frac{A_2 - k_1}{1-k_1}$ with

$$k_1 = \frac{(1+p-\delta)(2+p-\delta)(B-A)(B'-A')}{\left[\begin{array}{l}(p+1)(1+B)(1+B') + (1+p)(1+p-\delta)\{(1+B')(B-A) + (1+B)(B'-A')\} + \\ +(\delta-1)(1+p-\delta)^2 (B-A)(B'-A')\end{array}\right]}$$

PROOF: Proceeding exactly as in theorem (7.7.1), we require

$$\frac{\{(1+B_2)n+(B_2-A_2)(1+p-\delta)\psi(n,p,\delta)\}}{(B_2-A_2)}$$

$$\leq \frac{\{(1+B)n+(B-A)(1+p-\delta)\psi(n,p,\delta)\}}{(B-A)}$$

$$\frac{\{(1+B')n+(B'-A')(1+p-\delta)\psi(n,p,\delta)\}}{(B'-A')}$$

=C for all $n \geq 1$

That is

$$\frac{B_2-A_2}{1+B_2} \geq \frac{n\psi(n,p,\delta)}{\{c-(1+p-\delta)\}\psi(n,p,\delta)}$$

The function $\frac{n\psi(n,p,\delta)}{\{c-(1+p-\delta)\}\psi(n,p,\delta)}$ is decreasing with respect to n and so is maximum for n=1, we get

$$\frac{B_2-A_2}{1+B_2} \geq \frac{(1+p-\delta)(2+p-\delta)(B-A)(B'-A')}{\left[\begin{array}{l}(p+1)(1+B)(1+B')+(1+p)(1+p-\delta)\\ \{(1+B')(B-A)+(B-A)(B'-A')\}+(\delta+1)(1+p-\delta)^2\end{array}\right]} = K_1$$

Clearly K_1<1.Fixing in A_2 in (7.7.1),we get $B_2 \geq \frac{(A_2-k_1)}{1-k_1}$ as ,we requiring B_2<1-2k. Therefore $(f*g)(z) \in H(1-2k,1,p,\delta)$ with K_1 as in (7.7.7).

COROLLARY7.7.2 : Let $f,g,h \in H(A,B,p,\delta)$.

The $(f*g*n)(z) \in H(A_3,B_3,p,\delta)$,where $A_3 < 1-2k$ and $B_3 > \frac{(A_3+K_2)}{(1-k_2)}$

With $k_2 = \frac{(1+p-\delta)(2+p-\delta)(B-A)^2}{\left[(p+1)(1+B)^2+2(1+p)(1+p-\delta)(1+B)(B-A)+(\delta-1)(1+p-\delta)^2(B-A)^2\right]}$

THEOREM 7.7.3 Let the function f defined by (7.1.1), be in the $H(A,B,p,\delta)$ .Also le $g(z) = z_p - \sum_{n=1}^{\infty} |b_{p+n}| z^{p+n} \left(|b_{p+n}| \leq 1; p \in N\right)$.

Then $(f*g)(z)$ belongs to the class $H(A,B,p,\delta)$.

PROOF: Since

$$\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}|a_{p+n}||b_{p+n}|$$

$$\leq\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}|a_{p+n}|$$

$$\leq(B-A), \text{ by the theorem (7.2.1).}$$

Hence $(f*g)(z)$ belongs to class $H(A,B,p,\delta)$.

COROLLARY 7.7.3 Let the function f defined by (6.1.1) be in the class $H(A,B,p,\delta)$. Also let $g(z)=z^{p}-\sum_{n=1}^{\infty}|b_{p+n}|z^{p+n}\left(0\leq|b_{p+n}|\leq 1; p\in N\right)$.

Then $(f*g)(z)$ belongs to the class $H(A,B,p,\delta)$.

THEOREM7.7.4 Let the functions f and g defined by (5.1.1) and(5.1.2) respectively, belongs to the class $H(A,B,p,\delta)$ .Then

$$h(z)=z^{p}-\sum_{n=1}^{\infty}\left\{|a_{p+n}|^{2}+|b_{p+n}|^{2}\right\}z^{p+n},(p\in N)$$

Belongs to the class $H(A,B,p,\delta)$, where

$$A_4\leq 1-2k_3 and B_4\geq\frac{(A_4+k_3)}{(1-k_3)} with$$

$$k_3=\frac{2(1+p-\delta)(2+p-\delta)(B-A)^{2}}{\left[(1+p)\left\{(1+B)+(B-A)(1+p-\delta)^{2}-2(2+p-\delta)(1+p-\delta)^{2}(B-A)^{2}\right\}\right]}$$

PROOF: Since $f,g\in H(A,B,p,\delta)$. Then

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}|a_{p+n}|\leq 1$$

and

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}|b_{p+n}|\leq 1$$

$$\sum_{n=1}^{\infty}\left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}|a_{p+n}|\right]^2$$

$$\leq\left[\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}|a_{p+n}|\right]^2$$

$$\leq 1$$

Similarly

$$\sum_{n=1}^{\infty}\left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}|b_{p+n}|\right]^2\leq 1$$

Hence

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}|b_{p+n}|\right]^2\left\{|a_{p+n}|^2+|b_{p+n}|^2\right\}\leq 1$$

(7.7.8)

$h(z)\in H(A,B,p,\delta)$. if and only if

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{\{(1+B_4)n+(B_4-A_4)(1+p-\delta)\}\psi(n,p,\delta)}{(B_4-A_4)}|b_{p+n}|\right]^2\left\{|a_{p+n}|^2+|b_{p+n}|^2\right\}\leq 1$$

(7.7.9)

Comparing (6.7.9) and (6.7.8),we see that(6.7.9)is true if

$$\frac{\{(1+B_4)n+(B_4-A_4)(1+p-\delta)\}\psi(n,p,\delta)}{(B_4-A_4)}$$

$$\leq\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}|b_{p+n}|\right]^2=\frac{u^2}{2}$$

or

$$\frac{B_4-A_4}{1+B_4}\geq\frac{2n\psi(n,p,\delta)}{u^2-2\psi(n,p,\delta)(1+p-\delta)}=y(n)$$

(7.7.10)

Since $y(n)$ is decreasing functions with respect to n and the maximum for n=1. So (6.7.10) is satisfied

$$\frac{B_4-A_4}{1+B_4} \geq \frac{2(1+p-\delta)(2+p-\delta)(B-A)^2}{\left[(1+p)\left\{(1+B)+(B-A)(1+p-\delta)^2-2(2+p-\delta)^2(B-A)^2\right\}\right]}$$

$$=k_3$$

(7.7.11)

Keeping A_4 fixed in (6.7.11) ,We get

$$B_4 \geq \frac{(A_4-k_3)}{(1-k_3)} \text{ and } B_4<1 \text{ gives } A_4<1-2k_3$$

.Therefore $h(z) \in H(1-2k_3, 1, p, \delta)$ with K_3 as in(6.7.11).

7.8 CONTENTMENT RELATION:

With the aid of theorem (6.2.1), we immediately obtain the following theorems.

THEOREM 7.8.1 Let $0<\delta \leq 1, -1 \leq A_1 \leq A_2<1$ and $0 \leq B \leq 1$.

Then

$$H(A, B, p, \delta) \subset H(A_2, B, p, \delta).$$

THEOREM 7.8.2 Let $0<\delta \leq 1, - \leq A<1$ and $0 \leq B_1 \leq B_2 \leq 1$.

Then

$$H(A, B_1, p, \delta) \subset H(A, B_2, p, \delta).$$

COROLLARY 7.8.3 Let $0<\delta \leq 1, - \leq A_1 \leq A_2<1$ and $0 \leq B_1 \leq B_2 \leq 1$.

Then

$$H(A_1, B_2, p, \delta) \subset H(A_1, B_1, p, \delta) \subset H(A_2, B_1, p, \delta).$$

THEOREM 7.8.3 Let $0<\delta \leq 1, -1 \leq A \leq 1$ and $0 \leq B \leq 1$

Then

$$H(A, B, p, \delta)=H\left(\frac{1-B+2A}{1+B}, 1, p, \delta\right)$$

More generally, if $-1 \leq A' \leq 1$ and $0 \leq B' \leq 1$,then

$$H(A, B, p, \delta) \subset H(A', B', p, \delta)$$

If and only if

$$\frac{\{(1+B)n+(B-A)(1+p-\delta)\}}{(B-A)}=\frac{\{(1+B')n+(B'-A')(1+p-\delta)\}}{(B'-A')}$$

7.9 CLOSURE THEOREMS

THEOREM 7.9.1: Let $f_j(z)=z^p-\sum_{n=1}^{\infty}\left|a_{p+n,j}\right|z^{p+n}, j=1,2,3.....m; p\in N$ if $f_j\in H(A,B,p,\delta)$ for each j=1,2….,m then the function $h(z)=z^p-\sum_{n=1}^{\infty}\left|c_{p+n}\right|z^{p+n}$

Where $\left|c_{p+n}\right|=\frac{1}{m}\sum_{j=1}^{m}\left|a_{p+n,j}\right|$ also belongs to $H(A,B,p,\delta)$.

PROOF Since $f_j\in H(A,B,p,\delta)$. Then from the theorem6.2.1,we have

$$\sum_{n=1}^{\infty}(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}\left|a_{p+n,j}\right|\le(B-A) \quad \backslash$$

For each $j=1,2,...m$.

Therefore

$$\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}\left|c_{p+n}\right|$$
$$=\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}\frac{1}{m}\sum_{j=1}^{m}\left|a_{p+n,j}\right|$$

By the theorem (7.9.1)

Hence $h\in H(A,B,p,\delta)$.

THEOREM 7.9.2 The class $H(A,B,p,\delta)$ is convex.

PROOF: Let the functions f and g defined by (7.1.1) and (7.1.2), respectively ,be in the class $H(A,B,p,\delta)$ .Then it is sufficiently to show that the functions.

$$h(z)=\mu f(z)+(1-\mu)g(z) \qquad (0\le\mu\le1)$$

or equivalently

$$h(z)=z^p-\sum_{n=1}^{\infty}\left\{\mu\left|a_{p+n}\right|+(1-\mu)b_{p+n}\right\}z^{p+n}$$

$(0\le\mu\le1)$ is also the in the class $H(A,B,p,\delta)$.

Since $f,g\in H(A,B,p,\delta)$. Then from the theorem (7.2.1), we have

$$\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}\left|a_{p+n}\right|\le(B-A).$$

Therefore

$$\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}\{\mu|a_{p+n}|+(1-\mu)|b_{p+n}|\}$$

$$\mu\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}|a_{p+n}|$$

$$+(1-\mu)\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}|b_{p+n}|$$

$\leq(B-A)$, By theorem (7.2.1).

Hence $h\in H(A,B,p,\delta)$.

THEOREM 7.9.3: Let $f_p(z)=z^p$ and

$$f_{p+n}(z)=z^p-\frac{(B-A)z^{p+n}}{\{(1+B)n+(B-A)(n+p-\delta)\}\psi(n,p,\delta)}$$

Where

$-1\leq A<B\leq 1, 0<\delta\leq 1, p\in N.$

Then $f\in H(A,B,p,\delta)$ if and only if ,it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}c_{p+n}f_{p+n}(z)$$

where,

$$c_{p+n}\geq 0;\sum_{n=0}^{\infty}c_{p+n}=1.$$

PROOF: Let us suppose that

$$f(z)=\sum_{n=0}^{\infty}c_{p+n}f_{p+n}=z^p-\sum_{n=1}^{\infty}\frac{(B-A)c_{p+n}z^{p+n}}{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n.p,\delta)},$$

Where

$$c_{p+n}\geq 1;\sum_{n=0}^{\infty}c_{p+n}=1$$

Then

$$\sum_{n=1}^{\infty}\psi(n,p,\delta)\{(1+B)n+(B-A)(1+p-\delta)\}.$$

$$\left[\frac{(B-A)c_{p+n}}{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}\right]$$

$$=(B-A)\sum_{n=0}^{\infty}c_{p+n}=(B-A)(1-c_p)\leq(B-A)$$

Hence the theorem (7.2.1), $f \in H(A, B, p, \delta)$.It follows from the theorem (7.2.1), that

$$\left|a_{p+n}\right| \leq \frac{(B-A)}{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}, n \in N$$

Setting

$$c_{p+n} = \frac{\{(1+B)n+(B-A)(1+p-\delta)\}\psi(n,p,\delta)}{(B-A)}\left|a_{p+n}\right|, n \in N.$$

And $$c_p = 1 - \sum_{n=0}^{\infty} c_{p+n}.$$

We have $f(z) = \sum_{n=0}^{\infty} c_{p+n} f_{p+n}(z)$.

This is the complete proof of theorem.

CHAPTER 8

ANALYTIC FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVE HAVING TWO FIXED POINT

8.1 INTRODUCTION:

Let we T denote the functions of the form

$$f(z) = a_1 z^n - \sum_{n=2}^{\infty} a_n z^n, (a_1 > 0, a_n \geq 0)$$

(8.1.1)

Which are Analytic and Univalent in the unit disc $u = \{z : |z| < 1\}$.

If f and g are any two functions in the class T such that the function f is defined by (8.1.1) and

$$g(z) = b_1 z - \sum_{n=2}^{\infty} b_n z^n, (b_1 > 0, b_n \geq 0)$$

(8.1.2)

Then the Quasi-Hadamard Product of f and g is denoted by $f * g$ and it is defined by the power series

$$(f * g)(z) = a_1 b_1 - \sum_{n=2}^{\infty} a_n b_n z^n$$

(8.1.3)

A functions f belonging the to the class T is said to Star like Functions of order α and type of β if and only if

$$\left| \frac{\frac{zf''(z)}{f(z)} - 1}{\frac{zf''(z)}{f(z)} + (1 - 2\alpha)} \right| < \beta, z \in u,$$

(8.1.4)

For $0 \leq \alpha < 1$ and $0 < \beta \leq 1$ .We denote by T (α, β) the class of all starlike functions of order α and β, further more a function f belonging to the class T is said to convex functions of order α and type β if and only if $zf'(z) \in$ (α, β).We denote by C (α, β) the class of all convex functions of order a and the type β. .In particular for s =1, the class T (α, β)and C (α, β).Let $T_0$and T_1 be two subclasses of T consisting of functions f such that

$f(z_0)=z_0$ and $f'(z_0)=1$ for $0<z_o<1$, respectively. We denote by T (α,β,z_0), C (α,β,z_0), T_1 (α,β,z_0), C_1 (α,β,z_0), the classes obtained by taking intersections respectively of the classes T (α, β)and $C(\alpha,\beta)$ with T_i (i=0,1) that is

$$T_i(\alpha,\beta,z_0)=T_i(\alpha,\beta)\cap T_i(\alpha,\beta)\cap T_i(i=0,1) \quad (8.1.5)$$

And

$$C_i(\alpha,\beta,z_0)=C(\alpha,\beta)\cap T_i(\alpha,\beta)\cap T_i(i=0,1) \quad (8.1.6)$$

The class T_0 (α,β,z_0), C_0 (α,β,z_0), T_1 (α,β,z_0), and T_1(α,β,z_0), will studied.byGupta, and Ahmad, . $f(z)$ belonging to the class T_i (α,β,z_0) and C_i(α,β,z_0)(i=1,2).

We have introduced a new class $M(A,B,z_0,\delta,\mu)$ Analytic Functions defined by Fractional Derivative having two fixed points as defined below.

A function f is defined by (7.1.1) and satisfying

$$(1-u)\frac{f(z_o)}{z}+\mu f'(z_0)=1, 0<z_0<1, \quad (8.1.7)$$

is said to be in the class $M(A,B,z_0,\delta,\mu)$ if where $-1\le A<B\le 1, 0\le\mu\le 1$ and

$F^{\delta}(z)=\Gamma(2-\delta)z^{\delta-1}D_z^{\delta}f(z)$. Here $D_z^{\delta}f(z)$ denotes the Fractional Derivative of $f(z)$ of order δ is defined by (8.1.6).

Thus the condition (8.1.8) reduces, when A=(2α-β), $B=\beta,\delta=1$, to the inequality (8.1.4) and we have

$$M((2\alpha-1)\beta,\beta,z_0,1,0)= T_0(\alpha,\beta,z_0),$$

and

$$M((2\alpha-1)\beta,\beta,z_0,1,0)= T_1(\alpha,\beta,z_0),$$

In section (8.2), we have obtained the necessary and sufficient condition in terms of coefficient for a function f belonging to the class $M(A,B,z_0,\delta,\mu)$.In section 8.3, we have investigated the properties for the class $M(A,B,z_0,\delta,\mu)$.In sections 8.4.we have determined these class Preserving Integral Operator F is defined by (4.8.1) for the class $M(A,B,z_0,\delta,\mu)$.In sections 8.5, we have obtained the radius of convexity for the class $M(A,B,z_0,\delta,\mu)$.In sections 8.6,we have obtained the some results involving the Quasi-Hadamard Product of two functions to the class $M(A,B,z_0,\delta,\mu)$.In the sections 8.7,we have some contentment relations related to the class $M(A,B,z_0,\delta,\mu)$.In the sections 8.8 ,we have shown that the class $M(A,B,z_0,\delta,\mu)$ is closed under the Arithmetic Mean and Convex linear Combinations.

Note- Through out of this chapter, we assume that

$$\phi(\delta,n)=\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$$

8.2 NECESSARY AND SUFFICIENT CONDITION

THEOREM 8.2.1 : A function f defined by (8.1.1) belongs to the class $M(A,B,z_0,\delta,\mu)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(N+1)+(2-\delta)(B-A)\}a_n\le a_1(B-A)$$

(8.2.1)

Where

$$a_1=1+\sum_{n=2}^{\infty}(1-\mu+\mu n)a_n z^{n-1}$$

PROOF: Let $|z|=1$ .Then from(8.1.8),we have

$$\left|f^{\delta}(z)-F^{(\delta-1)}(z)\right|-\left|BF^{\delta}(z)-AF^{(\delta-1)}(z)\right|\le 0,$$

$$=$$

$$\left|-\sum_{n=2}^{\infty}\frac{(n-1)}{(n+1-\delta)}\phi(\delta,n)a_n z^{n-1}\right|-\left|(B-A)a_1+\sum_{n=2}^{\infty}\phi(\delta,n)\left\{\frac{B(n-1)+(B-A)(2-\delta)}{(n+1-\delta)}\right\}a^n z^{n-1}\right|$$

$$\le\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n+1)+(2-\delta)(B-A)\}a_n-a_1(B-A)$$

by the hypothesis.

Hence, by the Maximum Modulus Theorem, $f\in M(A,B,z_0,\delta,\mu)$.

Let we converse

$$\left|\frac{F^{\delta}(z)-F^{\delta-1}(z)}{BF^{\delta}(z)-F^{(\delta-1)}(z)}\right|$$

$$=\left|\frac{-\sum_{n=2}^{\infty}\frac{(n-1)}{(n+1-\delta)}\phi(\delta,n)a_n z^{n-1}}{(B-A)a_1+\sum_{n=2}^{\infty}\phi(\delta,n)\left\{\frac{B(n-1)+(B-A)(2-\delta)}{(n+1-\delta)}\right\}a_n z^{n-1}}\right|$$

$$<1, z\in u.$$

Since $|Re(z)|\le|z|$ for all z ,we have

$$Re\left|\frac{\sum_{n=2}^{\infty}\frac{(n-1)}{(n+1-\delta)}\phi(\delta,n)a_n z^{n-1}}{(B-A)a_1+\sum_{n=2}^{\infty}\phi(\delta,n)\left\{\frac{B(n-1)+(B-A)(2-\delta)}{(n+1-\delta)}\right\}a_n z^{n-1}}<1\right| \quad (8.2.2)$$

.

Choose the values of z on the real axis ,so that $\frac{F^{\delta}(z)}{F^{\delta-1}(z)}$ is real. Clearing the denominotor and letting of (8.2.2) and letting z=1 through the real values , we obtain

$$\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n-1)+(2-\delta)(B-A)\}a_n\le a_1(B-A).$$

This is the complete proof of the theorem.

COROLLARY 8.2.1 Let the functions f defined by(8.1.1) belong to the class $M(A,B,z_0,\delta,\mu)$ .Then

$$a_n\le\frac{(B-A)(n+1-\delta)}{\left[\phi(\delta,n)\{(1+B)(n-1)+(2-\delta)(B-A)\}-(1-\mu+\mu n)z_0^{n-1}(B-A)(n+1-\delta)\right]}$$

(8.2.3)

With inequality for

$$f(z)=\frac{\{(1+B)(n-1)+(2-\delta)(B-A)\}\phi(\delta,n)z-(B-A)(n+1-\delta)z^n}{\phi(\delta,n)\{(1+B)(n-1)+(2-\delta)(B-A)\}-(1-\mu+\mu n)z_0^n(B-A)(n+1-\delta)z_0^{n-1}}$$

(8.2.4)

8.3 DISTORTION THEOREM:

THEOREM 8.3.1 : Let the function f defined by (8.1.1) belongs to the class $M(A,B,z_0,\delta,\mu)$ .Then

$$a_1\left[|z|-\frac{(B-A)(2-\delta)(3-\delta)}{2\{1+B+(2-\delta)(B-A)\}}|z|^2\right]\le|f(z)|\le a_1\left[|z|+\frac{(B-A)(2-\delta)(3-\delta)}{2\{1+B+(2-\delta)(B-A)\}}|z|^2\right]$$

$, z\in u$ (8.3.1)

And

$$\frac{a_1}{\Gamma(2-\delta)}\left[|z|^{1-\delta}-\frac{(B-A)(3-\delta)}{2\{1+B+(2-\delta)(B-A)\}}|z|^{2-\delta}\right]$$

$$\le\left|D_z^{\delta}f(z)\right|\le\frac{a_1}{\Gamma(2-\delta)}\left[|z|^{1-\delta}-\frac{(B-A)(3-\delta)}{\{1+B+(2-\delta)(B-A)\}}|z|^{2-\delta}\right]$$

Where

$$a_1=1+\sum_{n=2}^{\infty}(1-\mu+\mu n)a_n z_0^{n-2}, z\in u.$$

PROOF: In view of equations (8.2.1) and $\phi(\delta,n)$ is non decreasing for, we

$$\frac{2}{(2-\delta)(3-\delta)}\{1+B+(2-\delta)(B-A)\}\sum_{n=2}^{\infty}a_n \tag{8.3.2}$$

$$\le\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n-1)+(2-\delta)(B-A)\}a_n\le a\le(B-A)$$

Which is equivalent to

$$\sum_{n=2}^{\infty}a_n\le\frac{a_1(B-A)(2-\delta)(3-\delta)}{2\{1+B+(2-\delta)(B-A)\}}$$

Consequently, we obtain

$$|f(z)|\ge a_1|z|-\sum_{n=2}^{\infty}a_n|z|^n\ge a_1|z|-|z|^n\sum_{n=2}^{\infty}a_n$$

$$\ge a_1\left[|z|-\frac{(2-\delta)(3-\delta)(B-A)|z|^n}{2\{1+B+(2-\delta)(B-A)\}}\right]$$

And

$$|f(z)|\le a_1|z|+|z|^2\sum_{n=2}^{\infty}a_n$$

$$\geq a_1|z|+|z|^2\sum_{n=2}^{\infty}a_n$$

$$\leq a_1\left[|z|+\frac{(2-\delta)(3-\delta)(B-A)|z|^n}{2\{1+B+(2-\delta)(B-A)\}}\right]$$

This is equivalent to (8.3.1).

Further, by using second inequality in (8.3.3), we observe that

$$\left|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)\right|\geq a_1|z|-\sum_{n=2}^{\infty}\phi(\delta,n)a_nz^n$$

$$\leq a_1|z|-|z|^2\sum_{n=2}^{\infty}\phi(\delta,n)a_n$$

$$\leq a_1\left[|z|-\frac{(3-\delta)(B-A)}{\{1+B+(2-\delta)(B-A)\}}\right]|z|^2$$

And

$$\left|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)\right|\leq a_1|z|+\sum_{n=2}^{\infty}\phi(\delta,n)a_nz^n$$

$$\leq a_1|z|+|z|^2\sum_{n=2}^{\infty}\phi(\delta,n)a_n$$

$$\leq a_1\left[|z|+\frac{(2-\delta)(B-A)}{\{1+B+(2-\delta)(B-A)\}}|z|^n\right]$$

This is equivalent to (8.3.2).

COROLLARY 8.3.1 : Let the function f defined by (8.1.1) belongs to the class $M(A,B,z_0,\delta,\mu)$. Then the function f is included in a disc with its centre at the origin and the radius R given by

$$\leq\frac{a_1}{\Gamma(2-\delta)}\left[1+\frac{(3-\delta)(B-A)}{\{1+B+(2-\delta)(B-A)\}}\right]$$

8.4 INTEGRAL OPERATOR:

THEOREM 8.4.1 : Let $c>-1$. If f belongs to the class $M(A,B,z_0,\delta,\mu)$. Then the function F defined by the(4.8.1)also belongs to the class $M(A,B,z_0,\delta,\mu)$.

PROOF: From the definitions (4.8.1) and (8.1.1) ,we have,

$$F(Z)=a_1z-\sum_{n=2}^{\infty}\frac{(c+1)}{(c+n)}a_nz^n$$

Therefore

$$\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n-1)+(2-\delta)(B-A)\}\left\{\frac{(c+1)}{(c+n)}\right\}a_n$$
$$<\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n-1)+(2-\delta)(B-A)\}a_n$$

$\leq a_n(B-A)$, by Theorem (8.2.1).

Hence, F belongs to the class $M(A,B,z_0,\delta,\mu)$.

COROLLARY 8.4.1 : If f belongs to the class $M(A,B,z_0,\delta,\mu)$. Then

$$F(z)=\int_0^z\frac{f(t)}{t}dt\in M(A,B,z_0,\delta,\mu).$$

8.5 RADIUS OF CONVEXITY :

THEOREM 8.5.1 Let the function f defined by (8.1.1) belongs to the class $M(A,B,z_0,\delta,\mu)$ .Then f is convex in the disc $|z|<r,$ where

$$r=inf\left[\frac{\phi(\delta,n)}{(n+1,\delta)n^2}\left\{\frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)}\right\}\right]^{\frac{1}{n-1}}$$

The result is sharp for the function given by (8.2.4).

PROOF: To show this result, it is sufficient to prove that

$$\left|\frac{zf''(z)}{f'(z)}\right|<1 \, for \, |z|<r$$

Now,

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq\frac{\sum_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{a_1-\sum_{n=2}^{\infty}na_n|z|^{n-1}}$$

Clearly

$$\left|\frac{zf''(z)}{f'(z)}\right|<,$$

If $\sum_{n=2}^{\infty} n^2 a_n |z|^{n-1} \le a_1$ (8.5.1)

By the theorem (8.2.1) ,we have

$$\sum_{n==2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{\frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)}\right\} a_n \le a_1$$

Hence (8.5.1) will holds, if

$$n^2 a_n |z|^{n-1} \le \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{\frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)}\right\}$$

or equivalently

$$|z| \le \left[\frac{\phi(\delta,n)}{n^2(n+1-\delta)} \left\{\frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)}\right\}\right]$$

Which proves the required result.

8.6 SOME RESULTS INVOLVING QUASI HADAMARD PRODUCT:

In the following theorems, we use the technique of Padmanabhan

THEOREM 8.6.1 Let the function f and g defined by (8.1.1) and(8.1.2) respectively ,be in the same class $f \in M(A,B,z_0,\delta,\mu)$ .Then $(f*g)(z)$ defined by (8.1.3) belongs to the class $f \in M(A,B,z_0,\delta,\mu)$.

Where $a_1 \le 1-2k$ and $B_1 \ge \frac{(A_1+k)}{(1-k)}$

with

$$k = \frac{(2-\delta)(3-\delta)(B-A)^2}{\left[2\{1+B+(2-\delta)(B-A)\}^2-(2-\delta)(3-\delta)(B-A)^2\right]}$$

The result is sharp.

PROOF: Since $f,g \in M(A,B,z_0,\delta,\mu)$. Then from the Theorem (8.2.1), we have

$$\leq \sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{ \frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)} \right\} \frac{a_n}{a_1} \leq 1$$

(8.6.1)

And

$$\leq \sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{ \frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)} \right\} \frac{b_n}{b_1} \leq 1 \qquad (8.6.2)$$

Where

$$a_1 = 1+\sum_{n=2}^{\infty}(1-\mu+\mu n)a_n z_0^{n-1} \quad \text{and}\, b_1 = 1+\sum_{n=2}^{\infty}(1-\mu+\mu n)b_n z_0^{n-1}$$

We find the values of A_1 and B_1 such that $-1 \leq A_1 < B_1 \leq 1$ for $(F*g) \in M \in (A_1, B_1, z_0, \delta, \mu)$.

Equivalently, we determine A_1, B_1 satisfying

$$\leq \sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{ \frac{(1+B_1)(n-1)+(2-\delta)(B_1-A_1)}{(B_1-A_1)} \right\} \frac{a_n b_n}{a_1 b_1} \leq 1$$

(8.6.3)

Combining (8.6.1) and (8.6.2) , we using the Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} u\sqrt{\frac{a_n b_n}{a_1 b_1}} \leq \left[\sum_{n=2}^{\infty} u\frac{a_n}{a_1}\right]^{\frac{1}{2}} \left[\sum_{n=2}^{\infty} u\frac{b_n}{b_1}\right]^{\frac{1}{2}}$$

$$\leq 1$$

Where

$$u = \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{ \frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)} \right\}$$

(8.6.3) is satisfied if

$$u_1 \frac{a_n b_n}{a_1 b_1} \leq u\sqrt{\frac{a_n b_n}{a_1 b_1}}$$

Where

$$u = \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{ \frac{(1+B_1)(n-1)+(2-\delta)(B_1-A_1)}{(B_1-A_1)} \right\}$$

But from (8.6.4) we get

$$\sqrt{\frac{a_n b_n}{a_1 b_1}} \leq \frac{1}{u}$$

Therefore it is enough to find u_1 such that

$$u_1 \le u^2$$

Or equivalently

$$u = \frac{\phi(\delta,n)}{(n+1-\delta)}\left\{\frac{(1+B_1)(n-1)+(2-\delta)(B_1-A_1)}{(B_1-A_1)}\right\}$$

$$\le \left[\frac{\phi(\delta,n)}{(n+1-\delta)}\left\{\frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)}\right\}\right]^2 = u^2$$

That is

$$\{(1+B_1)(n-1)+(2-\delta)(B_1-A_1)\}\phi(\delta,n) \le u^2(B_1-A_1)(n+1-\delta).$$

This yields

$$A_1 < \frac{\{u^2-\phi(\delta,n)\}(n+1-\delta)B_1-(n-1)\phi(\delta,n)}{u^2(n+1-\delta)-(2-\delta)\phi(\delta,n)} \quad (8.6.4)$$

It is easy to verify that

$$u^2(n+1-\delta)-(2-\delta)\phi(\delta,n)\ for\ \ n\ge 2.$$

Now the above inequality gives on simplification

$$\frac{B_1-A_1}{1+B_1} \ge \frac{(n-1)\phi(\delta,n)}{u^2(n+1-\delta)-(2-\delta)\phi(\delta,n)} \quad \text{for}\, n\ge 2.$$

(8.6.5)

The right hand side member decreases as n increases and it is maximum for n=2.

So (8.6.2) is satisfied.

$$\frac{B_1-A_1}{1+B_1} \ge \frac{(2-\delta)(3-\delta)(B-A)^2}{\left[2\{1+B+(2-\delta)(B-A)\}^2-(2-\delta)(3-\delta)(B-A)^2\right]} = k$$

(8.6.6)

Obviously k<1 and fixed A_1 in (8.6.6),we get

$$B_1 \ge \frac{A_1+K}{1-k}$$

Let $B_1=1$ then $A_1\le 1-2k$. Therefore $(f*g)(z)\in M(A,B,z_0,\delta,\mu)$ .with k is defined as (8.6.6).

The result is sharp for the function.

$$f(z)=g(z)=\frac{2\{1+B+(2-\delta)(B-A)\}z-(2-\delta)(3-\delta)(B-A)z^2}{2\{1+B+(2-\delta)(B-A)\}z-(1+\mu)(2-\delta)(3-\delta)(B-A)z_0}$$

COROLLARY 8.6.1 : Let the function f and g defined by (8.1.1) and (8.1.2) respectively, be in the class $M(A_1,B_1,z_0,\delta,\mu)$ .Then $(f*g)(z)$ defined by(8.1.3) belongs to the class $M(1-2k,1,z_0,1,\mu)$.Where

$$k=\frac{(B-A)^2}{(1+2B-A)^2-(B-A)^2}$$

THEOREM 8.6.2: Let the function f defined by (8.1.1) belongs to the class $M(A_1,B_1,z_0,\delta,\mu)$ and g defined by(8.1.2) belongs to the class $M(1-2k,1,z_0,1,\mu)$.Then $(f*g)(z)$ defined by (8.1.3) belongs to the class $M(A_2,B_2,z_o,\delta,\mu)$,where $A_2\le 1-2k_1$ and $B_2\ge\dfrac{A_2+k_1}{1-2k_1}$

with

$$K_1=\frac{(2-\delta)(3-\delta)(B-A)(B'-A')}{2\{1+B+(2-\delta)(B-A)\}\{(2-\delta)(B'-A')\}-(2-\delta)^2(3-\delta)(B-A)(B'-A')}$$

This result is possible for

$$f(z)=\frac{2\{1+B+(2-\delta)(B-A)\}z-(2-\delta)(3-\delta)(B-A)z^2}{2\{1+B+(2-\delta)(B-A)\}z-(1+\mu)(2-\delta)(3-\delta)(B-A)z_0}$$

And

$$g(z)=\frac{2\{1+B'+(2-\delta)(B'-A')\}z-(2-\delta)(3-\delta)(B'-A')z^2}{2\{1+B'+(2-\delta)(B'-A')\}z-(1-\mu)(2-\delta)(3-\delta)(B'-A')z_0}$$

PROOF: Proceeding exactly as in theorem (8.6.1), we require

$$\frac{\phi(\delta,n)}{(n+1-\delta)}\left\{\frac{(1+B_1)(n-1)+(2-\delta)(B_2-A_2)}{(B_2-A_2)}\right\}$$

$$\leq\left\{\frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)}\right\}\left\{\frac{\phi(\delta,n)}{(n+1-\delta)}\right\}^2\left\{\frac{(1+B')(n-1)+(2-\delta)(B'-A')}{(B'-A')}\right\}$$

=c for all $n\geq 2$

That is

$$\frac{B_2-A_2}{1+B_2}\geq\frac{(n-1)\phi(\delta,n)}{c(n+1-\delta)-(2-\delta)\phi(\delta,n)}$$

(8.6.7)

The right hand member decreases as increases and so is maximum for n=2.So (8.6.7) is satisfied.

$$\frac{B_1-A_1}{1+B_1}\geq\frac{(2-\delta)(3-\delta)(B-A)(B'-A')}{\left[2\{1+B+(2-\delta)(B-A)\}2\{1+B'+(2-\delta)(B'-A')\}-(2-\delta)^2(3-\delta)(B-A)(B'-A')\right]}$$

$=k_1$

(8.6.8)

Clearly $k_1<1$. Fixing A_2 in (8.6.8),we get

$$B_2\geq\frac{A_2+k_1}{1-k_1}\text{ , as we require}$$

$B_2\leq 1$,we immediately obtain $A_2\leq 1-2k_1$.Therefore,$(f*g)(z)$ belongs $M(1-2k_1,1,z_0,1,\mu)$ with k_1 as in (8.6.8).

COROLLARY 8.6.2 : Let $f,g,h\in M(A,B,z_0,\delta,\mu)$. Then $(f*g*h)(z)\in M(A_3,B_3,z_0,\delta,\mu)$ where $A_3\leq 1-2k_2$ and

$$B_3=\frac{(A_3+k_2)}{(1-k_2)}$$

$$k_2=\frac{(2-\delta)(3-\delta)(B-A)^2}{\left[2\{1+B+(2-\delta)(B-A)\}^2-(2-\delta)^2(3-\delta)(B-A)^2\right]}$$

The result is possible for

$$f(z)=g(z)=h(z)=\frac{2\{1+B+(2-\delta)(B-A)\}z-(2-\delta)(3-\delta)(B-A)z^2}{2\{1+B+(2-\delta)(B-A)\}-(1+\mu)(2-\delta)(3-\delta)(B-A)z_0}$$

THEOREM 8.6.3 : Let the function f defined by (8.1.1) be in the class $M(A,B,z_0,\delta,\mu)$. Also let $g(z)=b_1z-\sum_{n=2}^{\infty}b_nz^n\ (b_n\le 1)$.Then $(f*g)(z)$ belongs to the class $M(A,B,z_0,\delta,\mu)$.

PROOF:

$$\le\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n-1)+(2-\delta)(B-A)\}a_nb_n$$

$$\le\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n-1)+(2-\delta)(B-A)\}a_n$$

$\le a_1(B-A,)$ by the theorem (8.2.1)

Hence $(f*g)(z)$ belongs to the class $M(A,B,z_0,\delta,\mu)$.

COROLLARY 8.6.3 Let $f\in M(A,B,z_0,\delta,\mu)$.

Also let $g(z)=b_1z-\sum_{n=2}^{\infty}b_nz^n\ (0\le b_n\le 1)$ .Then $(f*g)(z)$ belongs to the class $M(A,B,z_0,\delta,\mu)$.

8.7 CONTAINMENT RELATION:

The proof of following theorems comes from the Theorems(8.2.1).

THEOREM 8.7.1 Let $0\le\delta\le 1, 0\le\mu\le 1, -1\le A_1\le A\le 1 and 0\le B_1\le B_2\le 1$.

Then

$M(A_1,B,z_0,\delta,\mu)>M(A_2,B,z_0,\delta,\mu)$.

THEOREM 8.7.2 Let $0\le\delta\le 1, 0\le\mu\le 1, -1\le A\le 1 and 0<A<B\le 1$.

Then

$$M(A_1,B_2,z_0,1,\mu)>M(A_1,B_2,z_0,\delta,\mu).$$

THEOREM 8.7.3 Let $0\le\delta\le 1, 0\le\mu\le 1, -1\le A\le 1 and 0\le B\le 1$.

Then $M(A,B_1,z_0,\delta,\mu)=M\left(\frac{1-B-2A}{1+B},1,z_0,\delta,\mu\right)$.

More generally, $-1 < A' < 1 and 0 \le B' \le 1$. Then

$M\left(A,B,z_0,\delta,\mu\right) = M\left(A',B',z_0,\delta,\mu\right)$, if and only if

$$\frac{(1+B)(n-1)+(2-\delta)(B-A)}{(B-A)} = \frac{(1+B')(n-1)+(2-\delta)(B'-A')}{(B'-A')}$$

8.8 CLOSURE THEOREMS:

THEOREM 8.8.1 $f_i(z) = a_{i,j}(z) - \sum_{n=2}^{\infty} a_{n,j} z^n,\ j = 1,2....$

And

$$h(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n$$

where

$$c_1 = \sum_{j=1}^{\infty} \lambda_j a_1 j, c_n = \sum_{j=1}^{\infty} \lambda_j a_1, j\left(n = 2,3...\right), \lambda_j \ge 0 and \sum_{j=1}^{\infty} \lambda_j = 1$$

If $f_j \in M\left(A,B,z_0,\delta,\mu\right)$ for each $j = 1,2...$, then $h \in M\left(A,B,z_0,\delta,\mu\right)$

PROOF: If $f \in M\left(A,B,z_0,\delta,\mu\right)$. Then we have the theorem (8.2.1) is that

$$\sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{(1+B)(n-1)+(2-\delta)(B-A)\right\} a_{n,j} \le a_{i,j}(B-A),\ j = 1,2...$$

Therefore

$$\sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{(1+B)(n-1)+(2-\delta)(B-A)\right\} a_{n,j}$$

$$\le \sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)} \left\{(1+B)(n-1)+(2-\delta)(B-A)\right\} \left\{\sum_{j=1}^{\infty} \lambda_j a_{n,j}\right\}$$

$< c_1(B-A)$, by the theorem (8.2.1)

THEOREM 8.8.2 Let $f_1(z) = z$

And

$$f_n(z) = \frac{\left\{(1+B)(n-1)+(2-\delta)(B-A)\right\}\phi(\delta,n)z-(B-A)(n+1-\delta)z^n}{\left\{(1+B)(n-1)+(2-\delta)(B-A)\right\}\phi(\delta,n)-(1-\mu+\mu n)-(n+1-\delta)(B-A)}$$

,n=2.3….

From the condition (8.1.7), Theorem (8.2.1) is satisfied. Hence f belongs to the class $M\left(A,B,z_0,\delta,\mu\right)$. Conversely, Let $f\in M\left(A,B,z_0,\delta,\mu\right)$ and $f\left(z\right)=a_1z-\sum_{n=2}^{\infty}a_nz^n$ .Then by the theorem (8.2.1),we have

$$a_n\leq\frac{\left(B-A\right)\left(n+1-\delta\right)}{\left\{\left(1+B\right)\left(n-1\right)+\left(2-\delta\right)\left(B-A\right)\right\}\phi\left(\delta,n\right)-\left(1-\mu-\mu n\right)\left(n+1-\delta\right)\left(B-A\right)z_0^{n-1}}$$

Setting we, have

$$\lambda_n=\frac{\left\{\left(1+B\right)\left(n-1\right)+\left(2-\delta\right)\left(B-A\right)\right\}\phi\left(\delta,n\right)-\left(1-\mu-\mu n\right)\left(n+1-\delta\right)\left(B-A\right)z_0^{n-1}a_n}{\left(B-A\right)\left(n+1-\delta\right)}$$

where n=2,3….

And

$$\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n$$

We have

$$f\left(z\right)=\lambda_1f_1\left(z\right)+\sum_{n=2}^{\infty}\lambda_nf_n\left(z\right)$$

This is the complete proof of the theorem.

CHAPTER 9.0

ANALYTIC FUNCTIONS DEFINED BY INTEGRAL PROPERTIES

9.1 INTRODUCTION:

A function f of $J(p)$ belongs to the class $J^*(A,B,p)$ if and only if

$$\left|\frac{\frac{zf'(z)}{pf(z)}-1}{\frac{Bzf'(z)}{pf(z)}-A}\right|<1, z\in u,$$

Where $-1<A<B<1$.The Class $J(A,B,p)$

. Now, we have investigate a new class $j(A,B,f,p,\delta)$ of analytic starlike function in terms of Fractional Integral Operator over the elements of $J^*(A,B,p)$ having negative coefficients.

A functions G belongs to the class $j(A,B,f,p,\delta)$,if it satisfies.

$$G(z)=\frac{\Gamma(1+p+\delta)}{\Gamma(1+p)}z^{-\delta}D_z^{-\delta}f(z), z\in u,$$

(9.1.1)

For some function f belonging to $j^*(A,B,p)$.

Here $D_z^{-\delta}f(z)$ denotes the Fractional Integral of $f(z)$ of order δ,defined by

$$G(z)=\frac{\Gamma(1+p+\delta)}{\Gamma(1+p)}z^{-\delta}D_z^{-\delta}f(z), z\in u,$$

Where f is Analytic Function in a simply connected region of z-plane containing the origin and multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta)>0$.

By giving the specific values of parameter p, we obtain the following sub classes studied by various researchers in earlier works.

(i) Taking p=1 in (9.1.1), the class $j(A,B,f,p,\delta)$ reduces to the class $j(k,p,f)$ for some $f\in j(p,k)$ studied by Kummer

(ii) Taking $\delta = 1$,the class $j\left(A,B,f,p,\delta\right)$ reduces the integral operator

$$G\left(z\right) = \frac{\left(1+p\right)}{z}\int_{0}^{z} f\left(\zeta\right)d\zeta$$

(iii) Taking p=1 and $\delta = 1$ in (9.1.1) in the class $j\left(A,B,f,p,\delta\right)$reduces to

$$G\left(z\right) = \frac{2}{z}\int_{0}^{z} f\left(\zeta\right),$$

The Integral Operator studied by Libera,. Since the operator defined by (9.1.1) may be treated as generalization of the Libera integral operator.

The present chapter is divided into ten sections for the systematic study if the class $j\left(A,B,f,p,\delta\right)$.Sections (9.2) provided Lemma (9.2.2) due to Goel, and Sohi, needed to prove the results of succeeding sections of this chapter. In section 9.3,we have obtained the necessary and sufficient conditions in terms of coefficients for the function G belonging to the class $j\left(A,B,f,p,\delta\right)$to the section9.4,we have obtained the contentment relation related to the class $j\left(A,B,f,p,\delta\right)$.In section 9.5,we have obtained the Class-Preserving Integral Operator of the form.

$$F(z) = \frac{\left(c+p\right)}{z^{c}}\int_{0}^{z} t^{c-1}G\left(t\right)dt \tag{9.1.2}$$

c>-p, for the class $j\left(A,B,f,p,\delta\right)$.In section 9.6 ,we have determined the radius of convexity for the class $j\left(A,B,f,p,\delta\right)$. In sections 9.7,we have found the Distortion Properties for the class $j\left(A,B,f,p,\delta\right)$.In Section 9.8 ,we have investigated some results involving Modified Hadamard Product of two functions belonging to the class $j\left(A,B,f,p,\delta\right)$.In section9.9, we have investigated the closure properties for the class $j\left(A,B,f,p,\delta\right)$.

9.2 PRELIMINARY LEMMA:

In this section, we state a lemma due to Goel and Sohi , S.N needed to prove the results of succeeding section of this chapter.

LEMMA 9.2.1 A function f defined by(6.1.1) belongs to class $J^{*}\left(A,B,p\right)$ if and only if

$$\sum_{n=1}^{\infty}\{(1+B)n+(B-A)p\}|a_{p+n}| \le (B-A)p. \qquad (9.2.1)$$

The result is sharp with the external function.

$$f(z)=z^p-\frac{(B-A)p}{\{(1+B)n+(B-A)p\}}z^n, n\in N. \qquad (9.2.2)$$

9.3 NECESSARY AND SUFFICIENT CONDITIONS

THEOREM 9.3.1 A function $G(z)=z^p-\sum_{n=1}^{\infty}|c_{p+n}|z^{p+n}$ belongs to the class $j(A,B,f,p,\delta)$ if and only if

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)}|c_{p+n}| \le 1 \qquad (9.3.1)$$

PROOF: By the definitions of function G belongs to the class $j(A,B,f,p,\delta)$,if it satisfies the relations (9.1.1) for some functions f belongs to $J^*(A,B,p)$. Let f defined by (6.1.1). Then a simple computation, we obtain

$$G(z)=\frac{\Gamma(1+p+\delta)}{\Gamma(1+p)}z^{-\delta}D_z^{-\delta}f(z)$$
$$=z^{-p}-\sum_{n=1}^{\infty}\frac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)}|a_{p+1}|z^{p+1}$$

Clearly

$$|c_{p+n}|=\frac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)}|a_{p+n}|$$

Or

$$|a_{p+n}|=\frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)}|c_{p+n}|, n\in N \qquad (9.3.\ 2)$$

The required result follows by using (9.3.3) in Lemma (9.2.1)

NOTE: Throughout this chapter we assume that

$$r(n,p,\delta)=\frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p-\delta)}$$

REMARK: Let $G \in j(A,B,f,p,\delta)$, where function f defined by(6.1.1)

$$G(z)=z^{p}-\sum_{n=1}^{\infty}\left|c_{p+n}\right|z^{p+n} \tag{9.3.3}$$

where

$$\left|c_{p+n}\right|=\frac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)}\left|a_{p+n}\right|$$

$$=\frac{\left|a_{p+n}\right|}{r(n,p,\delta)}$$

Clearly

$$\frac{\Gamma(n+1+p)\Gamma(1+p+\delta)}{\Gamma(1+p)\Gamma(n+1+p+\delta)}$$

$$=\frac{(1+p)(2+p).......(n+p)}{(1+p+\delta)(2+p+\delta)......(n+p+\delta).}$$

$$=\prod_{j=1}^{n}\frac{(j+p)}{(j+p+\delta)}<1 \text{ for all } \delta>0$$

Thus $\left|c_{p+n}\right|<\left|a_{p+n}\right|$, for all $n \geq 1$

And therefore.

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\left|c_{p+n}\right|$$

$$<\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}\left|a_{p+n}\right|$$

$\leq 1,$ since $f \in J^{*}(A,B,p)$

Hence $G \in j(A,B,f,p,\delta)$ and thus, we get the containment relation

$$J(A,B,f,p,\delta)\subset J^{*}(A,B,p) \tag{9.3.4}$$

Since

$$lim_{\beta\to 0}J(A,B,f,p,\beta)\equiv J^{*}(A,B,p)$$

The relations (9.3.4) can also be written as

$$J(A,B,f,p,\delta) \subset lim_{\beta \to 0} J(A,B,f,p,\beta).$$

9.4 CONTENTMENT RELATION:

THEOREM 9.4.1: If $0 < \beta \le \delta$, then $J(A,B,f,p,\delta) \subset J(A,B,f,p,\beta)$

PROOF: Let the function G defined by (9.3.3) belongs to the class $J(A,B,f,p,\delta)$.Then the theorem (9.3.1), we have

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)p\}\Gamma(1+p)\Gamma(n+1+p+\delta)}{(B-A)p\Gamma(n+1+p)\Gamma(1+p+\delta)} |c_{p+n}| \le 1.$$

Next, since $\beta \le \delta,$ we have

$$\frac{\Gamma(1+p)\Gamma(n+1+p+\beta)}{\Gamma(n+1+p)\Gamma(1+p+\beta)} \le \prod_{j=1}^{n} \left\{ \frac{j+p+\beta}{j+p} \right\} \text{ since } \beta \le \delta.$$

$$= \frac{\Gamma(1+p)\Gamma(n+1+p+\delta)}{\Gamma(n+1+p)\Gamma(1+p+\delta)}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)p\}\Gamma(1+p)\Gamma(n+1+p+\beta)}{(B-A)p\Gamma(1+p)\Gamma(1+p+\beta)} |c_{p+n}|$$

$$\le \sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)p\}\Gamma(1+p)\Gamma(n+1+p+\delta)}{(B-A)p\Gamma(1+p)\Gamma(1+p+\delta)} |c_{p+n}|$$

Using (9.4.1) in (9.4.1), we get

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)p\}\Gamma(1+p)\Gamma(n+1+p+\beta)}{(B-A)p\Gamma(1+p)\Gamma(1+p+\beta)} |c_{p+n}| \le 1$$

Hence $G \in j(A,B,f,p,\delta)$ is the complete proof of the theorem.

With the aid of theorem, we obtain the following theorems.

THEOREM 9.4.2 Let $0 \le \delta \le 1, -1 \le A_1 \le A_2 < 1$ and $0 \le B \le 1$.

Then $J(A_1,B,f,p,\delta) \subset J(A_2,B_1,f,p,\delta)$.

THEOREM 9.4.3 Let $0 \le \delta \le 1, -1 \le A < 1$ and $0 \le B_1 \le B_2 \le 1$.

Then $J(A,B_1,f,p,\delta) \subset J(A,B_2,f,p,\delta)$

COROLLARY 9.4.3: Let $0 \le \delta \le 1, -1 \le A_1 \le A_2 < 1$ and $0 \le B_1 \le B_2 \le 1$.

Then

$J(A_1,B_2,f,p,\delta)\subset J(A_1,B_2,f,p,\delta)\subset J(A_2,B_1,f,p,\delta)$

THEOREM 9.4.4: Let $0\le\delta\le 1, -1\le A<1$ and $0\le B\le 1$.

$$\text{Then } J\left(A,B,f,p,\delta\right)=J\left(\frac{1-B+2A}{1+B},1,f,p,\delta\right).$$

More generally, if $-1\le A'<1$ and $-1\le B'\le 1$.

$$\text{Then } J\left(A,B,f,p,\delta\right)\subset J\left(A',B',f,p,\delta\right).$$

If and only if

$$\frac{(1+B)+(B-A)p}{(B-A)p}=\frac{(1+B')+(B'-A')p}{(B'-A')p}$$

9.5 INTEGRAL OPERATOR:

THEOREM 9.5.1 Let C be a real number such that c>-p. If G belongs to $j\left(A,B,f,p,\delta\right)$,then the function F defined by (9.1.2) is also an element of $j\left(A,B,f,p,\delta\right)$.

PROOF: Let G defined by (9. 3.3).Then the definition of it is clear that

$$F(z)=z^p-\sum_{n=1}^{\infty}\left|d_{p+n}\right|z^{p+n}$$

Where

$$\left|d_{p+n}\right|=\left(\frac{c+p}{c+p+n}\right)\left|c_{p+n}\right|<c_{p+n}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}r(n,p,\delta)\left|d_{p+n}\right|$$

$$<\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}r(n,p,\delta)\left|c_{p+n}\right|$$

≤ 1, by theorem (9.3.1)

Hence $J\in\left(A,\ B,f,\ p\ ,\delta\right)$.

In the following theorem, we consider the converse problem of the above problem

THEOREM 9.5.2 Let $F(z)$ be a real number such that c>-p. If $F \in (A,B,f,p,\delta)$, then the function G defined in (8.1.2) is starlike in z<r

Where

$$r^* = \begin{matrix} inf \\ n \in N \end{matrix} \left[\frac{\{(1+B)n+(B-A)p\}(c+p)r(n,p,\delta)}{(B-A)(p+n)(c+p+n)} \right]$$

The result is sharp.

PROOF: Let $F(z) = z^p - \sum_{n=1}^{\infty} |c_{p+n}| z^{p+n}$. It follows then the form (9.1.2) that

$$G(z) = \frac{z^{1-c}\left[z^c F(z)\right]'}{(c+p)}$$

$$= z^p - \sum_{n=1}^{\infty} \left| \frac{(c+p+n)}{(c+p)} \right|$$

In order to established the required result, it suffices to show that

$$\left| \frac{zG'(z)}{G(z)} - p \right| < p \text{ for} \left| z < r^* \right|$$

Now

$$\left| \frac{zG'(z)}{G(z)} - p \right| \le \frac{\sum_{n=1}^{\infty} \frac{n(c+p+n)}{(c+p)} |c_{p+n}| |z^{p+n}|}{1 - \sum_{n=1}^{\infty} \frac{n(c+p+n)}{(c+p)} |c_{p+n}| |z^{p+n}|}$$

Thus

$$\left| \frac{zG'(z)}{G(z)} - p \right| < p \text{ if}$$

$$\sum_{n=1}^{\infty} \frac{(p+n)(c+p+n)}{(c+p)} |c_{p+n}| |z|^n < p$$

Since $F \in j(A,B,f,p,\delta)$,we have

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)p\}}{(B-A)} r(n,p,\delta) |c_{p+n}| \le p$$

Hence (9.5.1) will be satisfied if

$$\frac{(p+n)(c+p+n)}{(c+p)}|c_{p+n}||z|^n$$

$$< \frac{\{(1+B)n+(B-A)p\}}{(B-A)} r(n,p,\delta)|c_{p+n}|, \qquad \text{for each } n \in N$$

Or if

$$|z| < \left[\frac{\{(1+B)n+(B-A)p\}(c+p)r(n,p,\delta)}{(B-A)(p+n)(c+p+n)}\right]^{\frac{1}{n}}, \quad \text{for each}$$

$n \in N$

Therefore G is starlike $z < r^*$.

To show the sharpness of the result, we take

$$F(z) = z^p - \frac{(B-A)pz^{p+n}}{\{(1+B)n+(B-A)p\}r(n,p,\delta)}, n \in N$$

Clearly $F \in j(A,B,f,p,\delta)$ and thus

$$\left|\frac{zG'(P)}{G'(z)} - p\right| = \left|\frac{\dfrac{-n(c+p+n)(B-A)pz^n}{(c+p)\{(1+B)n+(B-A)p\}r(n,p,\delta)}}{\dfrac{(c+p+n)(B-A)pz^n}{(c+p)\{(1+p)n+(B-A)p\}r(n,p,\delta)}}\right|$$

$= p$ at $z = r*$

9.6 RADIUS OF P-VALENT CONVEXITY:

THEOREM 9.6.1 If $G \in j(A,B,f,p,\delta)$ .then G is p-valently convex in the disc $|z| < r^{**}$, where

The result is sharp.

$$\ddot{r} = \inf_{n \in N}\left[\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)(p+n)^2}\right]$$

PROOF: In order to obtain the required result, it is sufficient to show that

$$\left|1+\frac{G''(z)}{G'(z)}-p\right| \le p \quad \text{for } |z| < r^{**}.$$

Let G is defined by (9.3.3) .Then we have

$$\left|1+\frac{zG''(z)}{G'(z)}-p\right| \le \frac{\sum_{n=1}^{\infty} n(p+n)|c_{p+n}||z|^n}{p-\sum_{n=1}^{\infty} n(p+n)|c_{p+n}||z|^n}$$

There fore

$$\left|1+\frac{zG''(z)}{G'(z)}-p\right| < p \text{ if}$$

$$\sum_{n=1}^{\infty}\frac{(p+n)^2}{(p)^2}|c_{p+n}||z|^n < 1$$

(9.6.1)

But from Theorem (9.3.1), we have,

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}r(n,p,\delta)|c_{p+n}| \le 1$$

Therefore (9.6.1) will be satisfied if

$$\frac{(p+n)^2}{(p)^2}|c_{p+n}||z|^n < \frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)|c_{p+n}|}{(B-A)p}$$

For each n ∈N, or if

$$|z| < \left[\frac{\{(1+B)n+(B-A)p\}p.r(n,p,\delta)}{(B-A)(p+n)^2}\right]$$

For each n ∈N

Hence G is convex in $z < r^{**}$

To sharpness of result, we have

$$G(z) = z^p - \frac{(B-A)pz^{P+n}}{\{(1+B)n+(B-A)p\}r(n,p,\delta)}, n \in N$$

Then

$$\left|1+\frac{zG''(z)}{G'(z)}-p\right|$$

$$=\left|\frac{-\frac{(p+n)(B-A)pz^{p+n}}{\{(1+B)n+(B-A)p\}r(n,p,\delta)}}{p-\frac{(p+n)(B-A)pz^{n}}{\{(1+B)n+(B-A)p\}r(n,p,\delta)}}\right|$$

$$=p \text{ at } \quad z=r^{**}$$

This shows that the result is sharp.

9.7 DISTORTION THEOREM:

THEOREM 9.7.1 : If $G\in j(A,B,f,p,\delta)$ and $|z|=r$, where

$$r^{p}-\alpha r^{p+1}\le\left|G'(z)\right|\le pr^{p-1}+\alpha r^{p+1}$$

(9.7.1)

And

$$pr^{p-1}-\alpha(p+1)r^{p}\le\left|G'(z)\right|\le pr^{p-1}+\alpha(P+1)r^{p}$$

Where

$$\alpha=\frac{(B-A)p}{\{(1+p)+(B-A)p\}r(1,p,\delta)}$$

These inequalities are sharp.

PROOF: Let G is defined by (9.3.3).Then in view of theorem (9.3.1),we have

$$\frac{\{(1+B)+(B-A)p\}r(1,p,\delta)}{(B-A)p}\sum_{n=1}^{\infty}\left|c_{p+n}\right|$$

$$\le\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}r(n,p,\delta)\left|c_{p+n}\right|$$

$$\le 1$$

Which evidently yields

$$\sum_{n=1}^{\infty}\left|c_{p+n}\right|\le\frac{(B-A)p}{\{(1+B)+(B-A)p\}r(1,p,\delta)} \qquad (9.7.2)$$

Hence

$$\left|G(z)\right| \leq |z|^{p} + \sum_{n=1}^{\infty} |c_{P+n}||z|^{p+n}$$

$$\leq r^{p} + r^{p+1} \sum_{n=1}^{\infty} |c_{P+n}|$$

$$\leq r^{p} \frac{(B-A)pr^{p+1}}{\{(1+B)+(B-A)p\}r(1,p,\delta)}$$

and

$$\left|G(z)\right| \leq |z|^{p} - \sum_{n=1}^{\infty} |c_{P+n}||z|^{p+n}$$

Thus inequalities (9.7.1) hold.

Further

$$\left|G'(z)\right|^{2} \leq p|z|^{p-1} + \sum_{n=1}^{\infty} (p+n)|c_{p+n}||z|^{p+n-1} \qquad (9.7.3)$$

$$\leq pr^{p-1} + r^{p} \sum_{n=1}^{\infty} (p+n)(c_{p+n})$$

And

$$\left|G'(z)\right| \geq p|z|^{p-1} - \sum_{n=1}^{\infty} (p+n)|c_{p+n}||z|^{p+n-1}$$

$$\leq pr^{p-1} - r^{p} \sum_{n=1}^{\infty} (p+n)|c_{p+n}|$$

$$since$$

$$\frac{\{(1+B)+(B-A)p\}}{(B-A)p(p+1)} \sum_{n=1}^{\infty} (p+n)|c_{p+n}|$$

$$\leq \sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)p\}}{(B-A)p} r(n,p,\delta)$$

$$\leq 1$$

We have

$$\sum_{n=1}^{\infty} |p+n||c_{p+n}| \leq \frac{(B-A)p(P+1)}{\{(1+B)+(B-A)p\}r(1,p,\delta)}$$

The inequalities (9.7.2) follow now by the using (9.7.3) and (9.7.4)

The inequalities are obtained in (8.7.1) and(8.7.2) by taking

$$G(z) = z^{p} - \frac{(B-A)pz^{p+1}}{\{(1+B)+(B-A)p\}r(1,p,\delta)}, (z = \pm r)$$

COROLLARY 9.7.1 If $G \in j(A,B,f,p,\delta)$, then the disc u is mapped by G is onto a domain that contains the disc with center at the origin and radius 1^+ .The result is sharp.

9.8 SOME RESULTS INVOLVING MODIFIED HADAMARD PRODUCT:

In the following theorems, we use the technique of Padmanabhan

THEOREM 9.8.1 : If G(z) is defined by (9.3.3) and

$$H(z) = z^p - \sum_{n=1}^{\infty} |d_{p+n}| z^{p+n} \qquad ((9.8.1)$$

are elements of $j(A,B,f,p,\delta)$,then,

$$(G*H)(z) = z^p - \sum_{n=1}^{\infty} |c_{p+n}||d_{p+n}| z^{p+n} \qquad (9.8.2)$$

Is an element of $j(A,B,f,p,\delta)$, where $A_1 \le 1-2k$ and $B_1 \ge \frac{A_1+k}{1-k}$ with

$$k = \frac{(1+p)p(B-A)^2}{\{(1+B)+(B-A)p^2\}(1+p+\delta)-(1+p)p(B-A)^2}$$

PROOF: Since $G,H \in j(A,B,f,p,\delta)$. Then the theorem (9.3.1), we have

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)p\}}{(B-A)p} r(n,p,\delta)|c_{p+n}| \le 1 \qquad (9.8.3)$$

And

$$\sum_{n=1}^{\infty} \frac{\{(1+B)n+(B-A)p\}}{(B-A)p} r(n,p,\delta)|d_{p+n}| \le 1 \qquad (9.8.4)$$

We wish to fined the values A_1,B_1 such that -1<A_1>,$B_1 \le 1$, for (G*H)(z) belongs to $j(A,B,f,p,\delta)$.Equivalently, we want to determine A_1,B_1 satisfying.

$$\sum_{n=1}^{\infty}\frac{\{(1+B_1)n+(B_1-A_1)p\}}{(B_1-A_1)p}r(n,p,\delta)|c_{p+n}||c_{p+n}|\le 1$$

(9.8.6) Combining (9.8.3) and (9.8.4), we get using Cauchy-Schwarz inequality.

$$\sum_{n=1}^{\infty}u\sqrt{|c_{p+n}||d_{p+n}|}\le\left\{\sum_{n=1}^{\infty}u\sqrt{|c_{p+n}|}\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}u\sqrt{|d_{p+n}|}\right\}^{\frac{1}{2}}\le 1 \qquad (9.8.7)$$

Where

$u=\dfrac{\{(1+B)n+(B-A)p\}}{(B-A)p}r(n,p,\delta)$ for each $n\in N$,(9.8.6) is satisfied if

$$u_1=\{u_1|c_{p+n}||d_{p+n}|\}\le u\sqrt{|c_{p+n}||d_{p+n}|},$$

Where

$$u_1=\frac{\{(1+B_1)n+(B_1-A_1)p\}r(n,p,\delta)}{(B_1-A_1)p},(n\in N)$$

But from (9.8.6), we have

$$\sqrt{|c_{p+n}||d_{p+n}|}\le\frac{1}{u},(n\in N)$$

Therefore it is enough to find u_1 such that

$$\frac{1}{u}\le\frac{u}{u_1}$$

$$or$$

$$u_1\le u_2$$

Or equivalently

$$\frac{\{(1+B_1)n+(B_1-A_1)p\}}{(B_1-A_1)p}r(n,p,\delta)$$

$$\le\left[\frac{\{(1+B_1)n+(B_1-A_1)p\}r(n,p,\delta)}{(B_1-A_1)p}\right]^2$$

$$=u^2,n\ge 1$$

That is

$$\{(1+B_1)n+(B_1-A_1)p\}r(n,p,\delta)\le u^2(B_1-A_1)p$$

This yields

$$A_1 < \frac{u^2 B_1 - \frac{1}{p}\{B_1(n+p)+n\}r(n,p,\delta)}{u^2 r(n,p,\delta)}$$

It is easy to verify that $u_2 > r(n,p,\delta)$ for $n \ge 1$

Now, the above inequality gives by simple calculation

$$\frac{(B_1 - A_1)}{(1+B_1)} \ge \frac{nr(n,p,\delta)}{p\{u^2 - r(n,p,\delta)\}} \quad \text{For } n \ge 1 \tag{9.8.8}$$

The right hand member decreases as n increases and so its maximum for n=1 .So (9.8.7) is satisfied.

$$\frac{(B_1 - A_1)}{(1+B_1)} \ge \frac{(1+p)p(B-A)^2}{\{(1+B)+(B-A)p\}^2 r(n,p,\delta)-(1+p)p(B-A)^2} = k \tag{9.8.9}$$

Obviously k<1 and fixing A_1in the above inequality, we get

$B_1 \ge 1, then A_1 < 1-2k$

Therefore, $(G*H)(z) \in j(1-2k,1,f,p,\delta)$ with k is defined as(9.8.8).

COROLLARY 9.8.1 Let G is defined by (9.8.3) and H is defined by (9.8.1) are elements of $J(A,B,f,p,1)$, those $(G*H)(z)$is defined by (9.8.2) is an element of $J(A_1,B_1,f,p,1)$where $A_1 \le 1-2k$ and $B_1 \ge \frac{(A_1+k)}{(1-k)}$with

$$k = \frac{(1+B)p(B-A)^2}{(2+p)\{(1+B)+(B-A)p\}^2 - (1+p)p(B-A)^2}$$

THEOREM 9.8.2 Let G is defined by (9.8.3) belongs to the class $j(A,B,f,p,\delta)$ and H defined by (9.8.1) belongs to the class $j(A,B,f,p,\delta)$,then$(G*H)(z)$ defined by (9.8.2) belongs to the class $j(A_2,B_2,f,p,\delta)$, where $A_2 \le 1-2k_1$ and$B_2 \ge (A_2+k_1)/(1-k_1)$with

$$k_1 = \frac{(1+B)p(B-A)(B'-A')}{\begin{bmatrix}(1+p+\delta)(1+B)(1+B')+\delta p^2(1+B)(B'-A')+ \\ p(1+p+\delta)\{(1+B)(B'-A')+(1+B')(B-A)\}\end{bmatrix}}$$

PROOF: Proceeding exactly in the theorem (9.8.1),require

$$\frac{\left\{(1+B_2)n+(B_2-A_2)p\right\}}{(B_2-A_2)p}r(n,p,\delta)$$
$$\leq \frac{\left\{(1+B)n+(B-A)p\right\}}{(B-A)p}r(n,p,\delta).\frac{\left\{(1+B')n+(B'-A')p\right\}}{(B'-A')p}r(n,p,\delta)$$

=c for all n≥1

That is

$$\frac{B_2-A_2}{1+B_2}\geq \frac{nr(n,p,\delta)}{p\left[c-r(n,p,\delta)\right]}$$

The function $\frac{nr(n,p,\delta)}{p\left[c-r(n,p,\delta)\right]}$ is decreasing with respect to n and it is maximum for n=1, we get

$$\frac{B_2-A_2}{1+B_2}=\frac{(1+B)p(B-A)(B'-A')}{\begin{bmatrix}(1+p+\delta)(1+B)(1+B')+\lambda p^2(1+B)(B'-A')+ \\ p(1+p+\lambda)\left\{(1+B)(B'-A')+(1+B')(B-A)\right\}\end{bmatrix}} \quad (9.8.10)$$

$=k_1$

Clearly $k_1<1$

Fixing A_2 in(9.8.9),we get

$$B_2\geq (A_2+k_1)/(1-k_1)$$

As we require $B_2\leq 1$,we immediately obtain $A_2\leq 1-2k_1$.

Therefore $(G*H)(z)$ belongs to $j(1-2k_1,1,f,p,\delta)$ with k_1 as in (9.8.9)

COROLLARY 9.8.2 : Let $G,H,I\in j(A,B,f,p,\delta)$ .Then $(G*H*I)(z)\in j(A_3,B_3,f,p,\delta)$ where $A_3\leq 1-2k_2$ $B_3=(A_3+k_2)/(1-k_2)$.

With

$$k_2=\frac{p(1+p)(B-A)^2}{\left[(1+p+\delta)(1+B)^2+\lambda p^2(B-A)^2+2p(1+p+\delta)(1+B)(B-A)\right]}$$

THEOREM 9.8.3 : Let G defined by (9.3.3) belongs to the class $j(A,B,f,p,\delta)$.Also let $H(z)=z^p-\sum_{n=1}^{\infty}|d_{p+n}|z^{p+n},\left(|d_{p+n}|z^{p+n}\leq 1;\, p\in N\right)$.Then $(G*H)(z)$belongs to the class $j(A,B,f,p,\delta)$.

PROOF: Since

$$\sum_{n=1}^{\infty} r(n,p,\delta)\{(1+B)n+(B-A)p\}|c_{p+n}||d_{p+n}|$$

$$\leq \sum_{n=1}^{\infty} r(n,p,\delta)\{(1+B)n+(B-A)p\}|c_{p+n}|$$

$\leq (B-A)p$ by theorem (9.3.1.)

COROLLARY 9.8.3 : Let the function G be in the class $j(A,B,f,p,\delta)$.

Also let

$$H(z)=z^p-\sum_{n=1}^{\infty}|d_{p+n}|z^{p+n},\left(0\leq |d_{p+n}|;\, p\in N\right)$$

Then $(G*H)(z)$belongs to class $J(A,B,f,p,\delta)$.

THEOREM 9.8.4 : Let G defined by (9.3.3) and H defined by(9.1.1) belongs to the class $j(A,B,f,p,\delta)$.

Then

$$F(z)=z^p-\sum_{n=1}^{\infty}\left\{|c_{p+n}|^2+|d_{p+n}|^2\right\}z^{p+n}, n\in N$$

This belongs to the class $j(A_4,B_4,f,p,\delta)$, $A_4\leq 1-2k_3$ and $B_4\leq (A_4+k_3/1-k_3)$ .with

$$k_3=\frac{2p(1+p)(B-A)^2}{\left[(1+p+\delta)(1+B)^2+(\delta-1-p)p^2(B-A)^2+2p(1+p+\delta)(1+B)(B-A)\right]}$$

PROOF: Since $G,H\in j(A,B,f,p,\delta)$.

Then

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}r(n,p,\delta)|c_{p+n}|\leq 1$$

And

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}r(n,p,\delta)\left|d_{p+n}\right|\leq 1$$

Now

$$\sum_{n=1}^{\infty}\left[\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}\left|c_{p+n}\right|\right]^2$$

$$\leq 1$$

Similarly

$$\sum_{n=1}^{\infty}\left[\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}\left|d_{p+n}\right|\right]^2$$

Hence

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}\left[\left|c_{p+n}\right|+\left|c_{p+n}\right|\right]\right]^2$$

(9.8.11)

≤ 1 .If $j\left(A_4,B_4,f,p,\delta\right)$ if and only if

$$\sum_{n=1}^{\infty}\frac{\{(1+B_4)n+(B_4-A_4)p\}}{(B_4-A_4)p}r(n,p,\delta)\left[\left|c_{p+n}\right|^2+\left|d_{p+n}\right|^2\right]$$

(9.8.12)

$$\leq 1$$

Comparing (9.8.11) with (9.8.10) we have (9.8.11) is true if

$$\frac{\{(1+B_4)n+(B_4-A_4)p\}r(n,p,\delta)}{(B_4-A_4)p}$$

$$\leq\frac{1}{2}\left[\frac{\{(1+B_4)n+(B_4-A_4)p\}r(n,p,\delta)}{(B_4-A_4)p}\right]^2$$

$$=\frac{1}{2}u^2$$

Or

$$\frac{(B_4-A_4)}{(1+B_4)}\geq\frac{2nr(n,p,\delta)}{p\left\{u^2-2r(n,p,\delta)\right\}}$$

$$=y(n)$$

Since y(n) is decreasing function with respect to n and so it is maximum for n=1 . So (9.8.12) satisfied.

$$\frac{B_4 - A_4}{1+B_4} = \frac{2p\left(B-A\right)^2}{\left[\left(1+p+\delta\right)\left(1+B\right)^2 + p\left(B-A\right)p^2\left(\delta-1-p\right)+2p\left(1+B\right)\left(1+p+\delta\right)\left(B-A\right)\right]}$$
$$= k_3$$

Keeping A_4 fixed in (9.8.13), we get $B_4 \geq \left(A_4 + k_3\right)/1-k_3$ and $B_4 \leq 1$ gives $A_4 \leq 1-2k_3$ with k_3 as in (9.8.13).Therefore F belongs to $j\left(1-2k_3, 1, f, p, \delta\right)$ with k_3 as in (9 .8.13).

9.9 CLOSURE THEOREMS:

THEOREM 9.9.1 : Let $G_j\left(z\right) = z^p - \sum_{n=1}^{\infty}\left|c_{p+n,j}\right|z^{p+n}, \left(j=1,2....; p \in N\right)$. If G_j belongs to the class $S\left(A,B,f,p,\delta\right)$ for each class $j = 1.2........., m$, then the function $H\left(z\right) = z^p - \sum_{n=1}^{\infty}\left|d_{p+n}\right|z^{p+n}$, where $\left|d_{p+n}\right| = \frac{1}{m}\sum_{j=1}^{\infty}c_{p+n,j}$,also belongs to the class $j\left(A,B,f,p,\delta\right)$.

PROOF: Since $G_j \in j\left(A,B,f,p,\delta\right)$. Then the theorem (9.3.1), we have

$$\sum_{n=1}^{\infty}\frac{\left\{\left(1+B\right)n+\left(B-A\right)p\right\}}{\left(B-A\right)p}r\left(n,p,\delta\right)\left|c_{p+n,j}\right| \leq 1 \text{ For each j=1,2……,m}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\left\{\left(1+B\right)n+\left(B-A\right)p\right\}}{\left(B-A\right)p}r\left(n,p,\delta\right)\left|d_{p+n}\right|$$
$$= \sum_{n=1}^{\infty}\frac{\left\{\left(1+B\right)n+\left(B-A\right)p\right\}r\left(n,p,\delta\right)}{\left(B-A\right)p}\left\{\frac{1}{m}\sum_{j=1}^{m}\left|c_{p+n,j}\right|\right\}$$
$$= \frac{1}{m}\left[\sum_{j=1}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{\left\{\left(1+B\right)n+\left(B-A\right)p\right\}}{\left(B-A\right)p}r\left(n,p,\delta\right)\left|c_{p+n,j}\right|\right\}\right]$$

≤ 1 , by the theorem (9.3.1)

Hence, $H \in j\left(A,B,f,p,\delta\right)$.

THEOREM 9.9.2 The class $j(A,B,f,p,\delta)$ is convex.

PROOF: Let G and H defined by (9.3.3) and (9.8.1) respectively belongs to the class $j(A,B,f,p,\delta)$. Then it is sufficient to show that the functions

$F(z)=\mu(z)+(1-\mu)H(z),(0\le\mu\le1)$.

Or equivalently

$F(z)=z^p-\sum_{n=1}^{\infty}\left\{u\left|c_{p+n}\right|+(1-\mu)\left|d_{p+n}\right|\right\}z^{p+n}$,$(0\le\mu\le1)$ is also in the class $\left|c_{p+n}\right|$

.Then the theorem (9.3.1),we have

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}\left|c_{p+n}\right|\le1$$

and

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}\left|d_{p+n}\right|\le1$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}\left\{\mu\left|c_{p+n}\right|+(1-\mu)\left|d_{p+n}\right|\right\}$$

$$=\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}\left|c_{p+n}\right|+(1-\mu)\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}\left|d_{p+n}\right|$$

≤1

Hence F belongs to the class $j(A,B,f,p,\delta)$.

THEOREM 9.9.3 Let $G_p(z)=z^p$

And

$$G_{p+n}(z)=z^p-\frac{(B-A)p}{\{(1+B)n+(B-A)p\}r(n,p,\delta)}z^{p+n}$$

Where $G\in j(A,B,f,p,\delta)$ if and only if it can be expressed in the form

$$G(z)=\sum_{n=0}^{\infty}\alpha_{p+n}G_{p+n}(z),\ \text{where}\quad \alpha_{p+n}\ge0;\sum_{n=0}^{\infty}\alpha_{p+n}=1\ .$$

PROOF: Let us suppose that $G(z)=\sum_{n=0}^{\infty}\alpha_{p+n}G_{p+n}(z)$,

$$z^p - \frac{(B-A)p\alpha_{p+n}z^{p+n}}{\{(1+B)n+(B-A)p\}r(n,p,\delta)}z^{p+n}$$

Then

where

$$\alpha_{p+n} \geq 0 and \sum_{n=0}^{\infty}\alpha_{p+n} = 1$$

$$\sum_{n=1}^{\infty}\frac{\{(1+B)n+(B-A)p\}}{(B-A)p}r(n,p,\delta)\left[\frac{(B-A)p\alpha_{p+n}}{\{(1+B)n+(B-A)p\}}\right]$$

$$=\sum_{n=1}^{\infty}\alpha_{p+n} = 1-\alpha_p \leq 1$$

Hence, by the theorem (9.3.1), $G \in j(A,B,f,p,\delta)$. Conversely, let G belongs to $j(A,B,f,p,\delta)$. It follows from the theorem (9.3.1),that

$$|c_{p+n}| \leq \frac{(B-A)p}{\{(1+B)n+(B-A)p\}r(n,p,\delta)}, n \in N$$

Setting

$$\alpha_{p+n} = \frac{\{(1+B)n+(B-A)p\}r(n,p,\delta)}{(B-A)p}|c_{p+n}|$$

and

$$\alpha_p = 1-\sum_{n=1}^{\infty}\alpha_{p+n}$$

We have

$$G(z) = \sum_{n=0}^{\infty}\alpha_{p+n}G_{p+n}(z)$$

This is the complete proof of the theorem.

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CHAPTER 10

CONCLUSIONS AND FUTURE SCOPE

10.1 SUMMARY AND CONCLUSION

The study of Certain Sub C lasses of Analytic Function Related to Complex Order is one of the most fascaniciating aspect. The Analytic Function have been played very important role of Certain Sub Classes of Complex Order. The Study of my research work has been proposed to the study of function□.It is differentiabe to every point of □. Here the domain □ means a nonempty connected sub set of complex plane. It is a associated with symmetric conjugate point and defined in the open unit disc Where is a complex plane and mapped by the univalent function on the unit disc.

The purpose of my study is to present an alternative technique in which an explicit use is made of integral derivative to complex order. The relevant study of my research work made to easy. We described only those aspects of theory in the directions of which we have pursued the study further. In a number of cases, our approach is not only yields a generalizations of various known results but also give many new and refined and best estimates.

There have been many endeavors to the to the study of various views of Analytic Functions perspectives, both exhaustive studies in my research work from and ramble studies in journals and periodicals . A pursuit for Conjecture has not yet been taken up comprehensively and orderly through unverified attempts. It have been made to some there strains and stresses, but it deeper produces into the subject have not been made so far. It is many considered that the present investigation will be divulging and would dig out the several conjectures and several sub classes of univalent functions using different techniques such as convolution techniques, variation method and subordinate techniques. I also introduced the different families of certain sub classes such as $V(\lambda, v, A, B, b)$.The use of hyper geometric function in Bibeberbach conjecture has prompted renewed interest in classes of functions.

The study will further investigate injurious sprit of victim consciousness in Robertson conjecture. The tools and methodology used in research design in technique of Koebe univalent functions. Mapping properties of analytic functions.(H.P.F). Radius of p-valent convexity, distortion properties for the Class $Y(A, B, p, \delta)$ and preliminary for solving equations in different method for the estimations of the operator involved in the distortion theorem, integral operator developed by Goodmann,A.W and Closure theorems for finding numerical solutions of difference differential equations.

10.2 FUTURE SCOPE OF MY RESEARCH WORK :

The Analytic study of Certain Sub Classes Related To Complex Order is one of the most important areas of analysis and it is closely related to the very diverse areas of mathematics.. In the following years, the functional theory of complex variables developed widely. The development of various aspects of Analytic Function theory has been determined by basic research. The Certain classes of linear and nonlinear functions and Partial differential equations (PDE), optimization theory, control theory are important. The future scope of Certain Sub Classes Related To the Complex Order has support to find out the numerous applications in theoretical physics, mechanics and technology. The important problems in hydrodynamics and aerodynamics can be solved by using the methods of analytical study .