A STUDY OF FRACTIONAL CALCULUS ALONG WITH ITS SPECIAL FUNCTIONS AND THEIR APPLICATIONS IN ELECTRICAL, MECHANICAL AND PHYSICAL EQUATIONS

A THESIS

Submitted to wards the Requirement for the Award of Degree of

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS
Under the faculty of Science

BY

SUSHIL KUMAR DHANELIYA
(Enrollment No. 161510608914)

Under the Supervisor of

Dr. MANOJ SHARMA
Associate professor
Head, Dept. of Mathematics
RJIT BSF, Tekanpur Gwalior (M.P.)

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P.K. University, Shivpuri (M.P.)

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- Student must pass in Theory and Practical separately.
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- For pass the Ph. D candidate is required to obtain 65% marks in aggregate.

ABSTRACT:

The applications of fractional calculus can be seen in many areas. It has been played an important role in science and engineering. The abstract chapter wise given as under:

FIRST CHAPTER:

In this chapter, an introduction of Fractional Calculus and Special Functions with historical development have been given in detailed form. Fractional calculus acquired massive attentiveness of investigators during last five decades due to its multifarious applications in almost every discipline of science and engineering. Fractional calculus has now become the most prominent branch of mathematics. It is an extension of classical calculus. Presently it is established as one the most extensively used mathematical tool in diverse applications. Fractional calculus extends the derivatives of an integer order to an arbitrary order. In other words, fractional calculus is the field of mathematical analysis that deals with the investigation and applications of integrals and derivatives of arbitrary order. It is a natural generalization of classical calculus which extends the derivative of an integer order to an arbitrary order (real and complex). The credit for developing fractional calculus goes to L- Hospital and Leibnitz. On 30 September 1695

L- Hospital asked the question referring to the meaning of n "What if n is fractional ?" Leibnitz replied, "An apparent paradox from which one-day useful consequences will be drawn". These valuable words of Leibnitz became true after 300 years. Several eminent mathematicians namely Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov, Weierstrass, Heaviside, Mittag-Leffler, Feller, Erdelyi and Riesz etc. have made noteworthy contributions in the development of fractional calculus [1-2].Special functions are in general, real or complex-valued functions of one or more real or complex variables which are specified so completely that numerical values could in principle be tabulated [3]. Recently several special functions like Mittag-Leffler, Generalized M-series, M-function, Miller and Ross function, Wright function and Saxena's I-function have attained greater significance given their appearance as a solution

of various fractional differential equations. In particular, fractional calculus plays the most dominant role in dealing with the practical applications of special functions for solving various problems in fluid mechanics, heat conduction, quantum mechanics, kinetic equation, diffusion model, electrical engineering and electromagnetic waves, astrophysics etc [1-3].

In recent years, fractional calculus has become the most comprehensive mathematical technique in varied spheres of science and engineering.

Fractional calculus have been applied to generate enormous applications in numerous sectors of Biology, Physics, Electronics, Medical science, Economics and Finance, Electrical Engineering, Astrophysics etc. [1, 4-5].

In the present thesis, an attempt has been made to derive some theoretical applications of fractional calculus in the field of mechanical engineering, electrical engineering, and physics. We have introduced a fractional generalization of the standard kinetic equation and a new special function given by authors and also established the solution for the computational

extension of the Advanced fractional kinetic equation. Also, the 1-Dimensional fractal heat-conduction problem in a fractal semi-infinite bar has been developed by local fractional calculus (Calculus of arbitrary order) employing the analytical Advanced Yang-Fourier transforms method. Besides, we have obtained a solution of generalized Fractional integrodifferential equation of LCR circuit using hypergeometric series in terms of Mittag-Leffler function. In addition, we have obtained the closed-form solution of fractional differential equation associated with Newton's law of fractional order and fractional harmonic oscillator problem in terms of the Mittag-Leffler function.

SECOND CHAPTER:

In this chapter, The fractional calculus approach is applied in solving differential equation which is associated with an electrical circuit i.e. RLC circuit using hypergeometric series. The solution of the fractional differential equation of the RLC circuit comes in the form of the Mittag-Leffler function and the Ali et.al.[8] results are special cases of our main result.

The fractional integrodifferential equation with current on the capacitor is as

$$R\frac{d^{\alpha}}{dt^{\alpha}}i_c(t) + \frac{1}{c}i_c(t) + \frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}\, i_c(v)dv = \frac{d\theta(t)}{dt} \quad \text{--(1)}$$

The solution is as under:

$i_c(t) =$

$$\frac{1}{R}\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\{t^{\alpha+k-1}\sum_{r=0}^{\infty}(-a)^r\, t^{\alpha r} E_{2\alpha,\alpha+k+\alpha r}^{r+1}(-bt^{2\alpha})\} \quad \ldots (2)$$

THIRD CHAPTER:

In this chapter, We have introduced a fractional generalization of the standard kinetic equation and a new special function given by authors and also established the solution for the computational extension of the Advanced fractional kinetic equation. The results of the computational extension Advanced generalized fractional kinetic equation and its special cease are the same as the results of Chourasia and Panday [17] (2010)

Advanced Generalized Fractional Kinetic Equations:

In this chapter, we investigate the solution of the advanced generalized fractional kinetic equation. The results are obtained in a compact form in terms of modified Generalized $\mathcal{M}-$ function. The result ispresented in the form of a theorem as follows:

Theorem 1:

If $b \geq 0,\ c > 0,\ \alpha > 0,\ \beta > 0,\ \gamma > 0,\ \delta > 0,\ \rho > 0$ and $(\gamma\alpha - \beta) > 0$ than for the solution of the Advanced generalized fractional kinetic equation

$$N(t)\text{-}\ N_0\, {}_p^{\alpha,\beta,\gamma,\delta\tau,\rho}\mathcal{M}_q^{-c^{\alpha},b_1,\ldots b_n;b}(t) = -\sum_{r=1}^{n}\binom{n}{r} c^{r\alpha} D_t^{-r\alpha}\, N(t) \ldots (3)$$

Then

$$N(t) = N_0 {}^{\alpha,\beta,(\gamma+n),\delta\tau,\rho}{}_{p}\mathcal{M}_{q}^{-c^{\alpha},b_1,\ldots b_n;b}(t) \quad \ldots(4)$$

FOURTH CHAPTER:

In this chapter, we presented an analytical solution of 1-Dimensional heat conduction in the fractal semi-infinite bar by the Advanced Yang-Fourier transform of non-differentiable functions. The above findings are very useful in solving the practical problems because we have applied a partial fractional differential equation on a Cantor set.

$$T(x,t) = (Mf)(x) = \frac{\Gamma(1+\alpha)}{4^{\alpha} t^{\frac{\alpha}{2}} \pi^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} f(\xi) S_{\alpha}\left(-\frac{(x-\xi)^{2\alpha}}{4^{\alpha} t^{\alpha}}\right) (d\xi)^{\alpha}$$

... (5)

FIFTH CHAPTER:

In this chapter, we first presented a fractional derivative operator, which is also a generalization of truncated M-fractional derivative, by using generalized S-series. Then we defined the corresponding integral operator. Unlike fractional operators with different kernels, we showed that there are many common properties provided by both these and the corresponding integer-order operators. We also used these operators in differential equation problems as applications These problems are hard to solve using the classical definitions of fractional derivatives. Besides, from equality (e) of Example 1, we observed that, for polynomials, truncated

M-series fractional derivative coincides with the Riemann-Liouville and Caputo fractional derivatives [20] up to a constant multiple. In this case, we can say that the truncated S-series fractional derivative operator can be used instead of Riemann-Liouville or Caputo type derivatives (and also their generalizations) to solve some difficult problems. Our definition is also a generalization of the M-fractional derivative for p = q = 1 which defined in [38]. It is also possible to define new fractional derivatives by using other special functions instead of S-series. Since S-series is a general class of special functions, all future definitions have a chance to be the special cases of our definition.

$$C_{\alpha,1}(x) = \frac{1}{2}\left[E_{\alpha,1}(ix) + E_{\alpha,1}(-ix)\right],$$

$$S_{\alpha,1}(X) = \frac{1}{2}\left[E_{\alpha,1}(ix) - E_{\alpha,1}(-ix)\right] \quad \ldots (6)$$

SIXTH CHAPTER:

This chapter provides a basic introduction to fractional calculus (Calculus of arbitrary order). Here, we consider two different definitions of the fractional derivative, namely the Riemann-Liouville and Caputo forms. Later, we discuss fractional mechanics, where the time derivative is replaced with a fractional derivative of order α. We then solve some simple fractional differential equations of mechanics.

SEVENTH CHAPTER:

In this chapter, a new approach of the derivative of arbitrary order (FD) with the kernel of the smooth type that gains different depictions for the temporal and spatial variables has been given. It first applies to the time variables and hence it is fit to use transform of Laplace type (LT). Secondly, a definition is linked to the spatial type variables, by a global derivative of arbitrary order (FD), for which we will apply the transform of Fourier type (FT). The courtesy for this new methodology with a kernel of regular type was native from the vision that there is a period of global systems, which can designate the material heterogeneities and the fluctuations of unlike scales, which cannot be well described by traditional local theories or by arbitrary order models with the kernel of singular type.

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Dedicated

To

My Mother and Father

Smt. Ram Dhaneliya

Shri Narendra Kumar Dhaneliya

Senior Auditor (Defence)

Table of Content

CHAPTER-1

PRELIMINARIES

1.1 INTRODUCTION:

The fractional calculus is a generalization of integer order differentiation to non-integer cases. In other words, the fractional calculus operators deal with integrals and derivatives of arbitrary (i.e. real or complex) order. The name "fractional calculus" is a contradiction; the designation, "integration and differentiation of arbitrary order" is more fit. The traditional calculus was independently revealed in the seventeenth century by Sir I. Newton and G. W. Leibnitz. The question raised to Leibnitz by L'Hospital about the reality of fractional derivative of order half was a continuing topic amongst mathematicians for more than three hundred years, hence numerous aspects of fractional calculus were established and considered. Fractional calculus can be well-thought-out as a union of special functions and integral transforms. The special functions of mathematical physics can be considered as generalized fractional integrals or derivatives of basic elementary functions x^t, e^xetc. Also, numerous generalized Laplace-type integral transforms can be seen as transformation operators (which are fractional integrals) of the Laplace transform. Special roles in the applications of fractional calculus operators are played by the transcendental functions like the Mittag-Leffler function, M-series, k_2-function,k_4-function, Miller-Ross function, Wright's functions, and more generally Meijer's G-functions, Fox's H-functions, Saxena's I-function, and Südland, Baumann, and Nonnenmacher's $\aleph$ −aleph function.

Throughout the last decade applied mathematicians and physicists found the fractional calculus operators to be valuable in a variety of fields such as biology, chemistry, scattering theory, probability theory, control theory, potential theory, signal processing, image processing, diffusion theory, kinetic theory, heat transfer, and circuit theory, etc. The fractional calculus operators also occur extensively in technical problems associated with transmission lines and the theory of compressional shock waves.

The first precise use of a derivative of non-integer order is due to the French mathematician S. F. Lacroix [149] in 1819 who expressed the derivative of non-integer order ½ in terms of Legendre's factorial symbol Γ.

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt$$

Starting, with a function $y = x^p$, Lacroix expressed it as follows

$$\frac{d^n y}{dx^n} = \frac{p!}{(p-n)!} x^{p-n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}$$

Replacing with $\frac{1}{2}$ and putting $p = 1$, he obtained the derivative of order $\frac{1}{2}$ of the function x.

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = \frac{2}{\sqrt{\pi}} \sqrt{x}$$

In 1822, J. B. J. Fourier made the following integral representation

$$\frac{d^u f(x)}{dx^u} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha) d\alpha \int_{-\infty}^{+\infty} p^u \cos\left(px - p\alpha + \frac{u\pi}{2}\right) dp$$

where the number u was regarded as any quantity so forth, positive or negative.

The recognition of the first application of fractional calculus goes to Abel [2] who worked it in the solution of an integral equation which appeared in the formulation of the tautochrone problem of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire to the lowest point of the wire is the same regardless of the position of the bead on the wire.

Abel's [2] solutions involved a group of mathematicians and scientists in this branch of knowledge and the first logical definition of fractional derivative was given by Riemann−Liouville. Afterward, numerous attempts were made to define diverse forms of fractional integral and derivatives. Some of the important contributions are made by Weyl [325], Erdelyi [66], and Kober [142]. Alternatively, numerous applications of the calculus of fractional order were required by various mathematicians, engineers, and scientists. The efforts were so worthwhile that the subject of fractional calculus itself was considered applicable mathematics.

1.2 Historical Development and Review of Work Already Done:

Fractional calculus like many other mathematical disciplines and concepts has its origin in the motivation for the extension of meaning. One cannot study fractional calculus without provident the names of most of the great mathematicians of the world and feeling honored for their successes and some sense of the most important development, during the past four centuries and twenty years.

The Seventeenth Century:

The notation $d^n y/dx^n$ for n^th derivative was presented by Gottfried Wilhelm Leibnitz [155] during the budding period of calculus. In 1695, while answering a query of a French mathematician Guillaume died at L' Hospital regarding the possibility of n being a fraction, he answered that this deceptive paradox would one day be tackled and consequent developments would be amazing. In 1697,

Gottfried Wilhelm Leibnitz used the notation $d^{1/2}$ while referring to Wallis's infinite product for $\pi/2$ and detailed that differential calculus might have been used to achieve the same result.

The Eighteenth Century:

In 1730 Euler cited interpolation between integral orders of a derivative. Lagrange [150] underwrote by introducing the law of exponents for operators of integers order. Afterward, when the theory of fractional calculus began to take shape, it became significant to as certain that this law held correct for an arbitrary order derivative.

The Nineteenth Century:

The nineteenth century is considered the golden age of fractional calculus. Many prominent mathematicians developed this theory in this era. The first major study of fractional calculus began in 1832 when Liouville [158] considered $\frac{d^{1/2}}{dx^{1/2}}\,(e^{2x})$ and solved some problems in mechanics and geometry by the use of fractional operators. Riemann [244] in 1847 considered$\frac{d^{1/2}}{dx^{1/2}}$ (e^{2x}) and solved some problems in mechanics and geomet ry by the use of fractional operators. Riemann [244] in 1847 defined fractio nal integration by a generalization of a Taylor series expansion and contribu ted the following definition for fractional integration.

$$\frac{d^{-p}}{dx^{-p}}u(x) = \frac{1}{\Gamma(\mathrm{p})}\int_{c}^{x}(x-k)^{p-1}\,u(k)dk$$

He also added a complementary function in the above definition. owadays, this definition is in common use as a definition for

fractional integration but the complementary function is taken to be id entically zero with the lower limit of integration c replaced by zero.

Nevertheless in 1848 a famous mathematician Hargreave [95] general ized Leibnitz's rule for the n^{th} derivative of a product to β^{th}derivative, β bei ng arbitrary. In 1868 Letnikov [156] solved certain differential equations by the theory of fractional calculus and also discussed the effort of Liouville, Peacock, and Kelland.

In the growth of the definition of fractional derivative numerous math ematicians such as Sonin (1869), Letnikov (1872), Laurent (1884), Nekraso ve (1888), Nishimoto (1987) contributed tothe development of various conc epts and properties of fractional calculus.

The Twentieth Century:

The opening of the 20th century witnessed additional growth in this disciplin e. In 1927, Davis [50] suggested the notation ${}_{c}D_{x}^{-\beta}f(x)$ to define fractional i ntegration as

$$\frac{1}{\Gamma(\beta)}\int_{c}^{x}(x-t)^{\beta-1}f(t)dt$$

In 1939, Erdelyi [66] gave transformation of hypergeometric integrals using fractional integration by parts. Far ahead in 1940 Kober [142] extended some results of Hardy and Littlewood over a wider range, dealt with Mellin transforms, and also established auniqueness theorem for a solution to the equation

$$g(x)=\int_{a}^{x}(x-t)^{\beta-1}f(t)dt$$

Subsequently, in 1941 Widder [326] attempted to join the Laplace transform with fractional integrals.

Zygmund [346] derived several theorems on fractional derivatives, in 1945. Riesz [245] applied fractional integration to the theory of Riemann equation, relativistic theory, wave equation, and potentials in 1949. In 1950, Stuff [312] attained a relation on differences of fractional order as under,

$$\triangle^{\beta}x_{n}=\sum_{v=0}^{\infty}(-1)^{v}\binom{\beta}{v}x_{n+v}$$

Subsequently, In 1960 Erdelyi and Sneddon [70] studied fractional integration of dual integral equations. In 1964, Buschman [28] shortened the dual integral equation to a single integral equation by using fractional integral operators. In 1965, McCollum and Brown [189] made a noteworthy contribution by giving a list of Laplace and Inverse Laplace transforms related to fractional order calculus. These lists facilitated in finding the solution of fracti

onal differential equations. In 1969, an amazing contribution has been made
by Agarwal [8] to extend fractional calculus into fractional q-calculus.

The 1970's observed a huge gush in activities in the area of fractional
calculus and its applications. To instigate with Osler [224] studied certain g
eneralizations of the Leibnitz rule for the derivative of the product of two fu
nctions and used them to generate several infinite

series expansions relating special functions. ubsequently, he defined t
he fractional derivative by generalizing the Cauchy integral formula. These
generalizations allowed him to evaluate the value of a hypergeometric functi
on of unit argument in terms of Legendre’s gamma function. Oldham and S
panier [220] worked on the replacement of Fick’s laws by a formulation inv
olving semi-differentiation in 1970 itself.

Love [162] obtained two index laws for fractional integrals and deriv
atives in 1972. Also, Oldham and Spanier [221] developed a general solutio
n of the diffusion equation for semi-infinite geometries.

During the second half of the century, the acclaim of the growth of fr
actional calculus goes to Bertram Ross [247]. He has also organized the Firs
t International Conference on Fractional Calculus and its Applications at the
University of New Haven in June 1974 and edited its proceedings.

In 1974, K.B. Oldham and J. Spanier [222], published a book devoted
to fractional calculus. They have given the historical development of fractio
nal calculus with problems of mass and heat transfer in terms of the so-
called semi-derivatives and semi-integrals.

In the same year, Diaz and Osler [62] defined differences of fractio nal order and derived a Leibnitz rule for the fractional difference of the prod uct of two functions.

The energy received in the 1970s continued in the 1980s also. Saxena and Modi [271] dealt with certain multidimensional fractional integral oper ators associated with Gauss hypergeometric functions and

provided three theorems for these operators which provide expression s for their Mellin transforms and integration by parts. In 1984, Raina [241] obtained a fractional derivative of a general system of polynomials. Al-Bassam [12] studied the utilization of the fractional calculus method in solvi ng some classes of differential equations of Hermit's type and gave an appli cation of fractional calculus to differential equations in 1985. Also, in the sa me year, Saxena and Modi [272] defined certain fractional q −integral oper ators associated with a basic analog of Srivastava-Daoust's function.

In 1990, Saxena and Ram [277] introduced certain multidimensional Kober operators. Also, Gupta and Agrawal [92, 93] offered a correlation bet ween Dirichlet averages and fractional derivatives. Nonnenmacher provided an application of fractional calculus to a class of Levi distribution functions . Nishimoto [211] provided an exhaustive treatment on fractional calculus ra nging over four volumes in 1990. Furthermore, in 1992, Goyal, Jain, and Ga ur [54] considered fractional integral operators involving a product of gener alized was made geometric functions and a general class of polynomials.

In the last decade of the 20^{th} century, countless contribution was mad e by Miller and Ross [195] by publishing a book on fractional calculus and fractional differential equation.In 1993, Tuan and Saigo [320] gave some new multi-dimensional operators of fractional calculus, considered in certain spaces of generalized functions. They have applied these operators to elementary and generalized hypergeometric functions of multivariable. Again, Saxena and Singh

[282] introduced two new fractional integration operators associated with I-function in 1993. In the same year,Deora and Banerji [54, 55] established some results of double and triple Dirichlet averages by using fractional calculus.

In 1994, Saxena, Kiryakova, and Dave [267] attempted to unify and extended several results on fractional integral operators by taking up a layer of new fractional-order integral operators. During the same year, Deora and Banerji [56] gave an application of fractional order calculus to the solution of the Euler-Darboux equation in terms of Dirichlet averages. Podlubny [233] employed fractional calculus in control theory and Westerlund and Ekstam [324] gave the application of fractional calculus for developing capacitor theory during the same period. In 1995, Rutman [250] developed a physical interpretation of fractional integration and differentiation. In 1996, Mainardi [171] solved fractional relaxation-oscillation and fractional diffusion-wave phenomena. Kulkarni, Naikh, and Srivastava [145] gave an application of fractional calculus in solving a new class of multivalent functions with negative coefficients during the same period. In 1999, Jain and Jain [114] employed fractional order integral operators to solve some dual integral equat

ions involving I-
function and obtained relation in terms of finite sums of integrals involving H-function in 1999.

At the end of the century, fractional calculus was familiar to every an alyst and was part of the mathematics curriculum in the universities.

The Twenty-First Century:

Until contemporary times, fractional calculus was measured as a rather obsc ure mathematical theory without applications, but in the last decade, there h as been an explosion of research activities the application of fractional order calculus operators to different scientific fields such as fractional control of engineering systems, advancement of the calculus of variations and optimal control to fractional dynamic systems, analytical and numerical tools and te chniques, electrical and thermal constitutive relations, fundamental understa nding of wave and diffusion phenomenon, their measurements and verificati ons, thermal modeling of engineering systems such as brakes and machine t ools, Image and signal processing and bioengineering applications, etc.At th e beginning of the present century, Hilfer [101] summarized applications of fractional calculus in physics in his treatise in the year 2000. Also,Mainardi, Roberto, Gorenflo, and Scalas [178] gave the theory of tick-by-
tick dynamics of financial markets based on a continuous-
time random walk model and pointed out its consistency with the behavior o bserved in the waiting-
time distribution for Bund future prices traded at LIFFE, London by using fr actional calculus and continuous-
time finance. Haubold and Mathai [99]generalized simple kinetic equations used in astrophysics to a fractional kinetic equation and obtained its solution in terms of H-

function and emphasized the role of thermonuclear function which are represented in terms of G-and H-functions. Again, Srivastava and Saxena [311] presented a systematic and historical account of the investigations carried out by variousauthors in the field of fractional calculus and its applications, thus providingan effective tool for understanding the subject. Jain and Pathan [115] established several

theorems involving Laplace transform and Weyl fractional integral operators and applied them in finding a large number of useful results in 2001. Ali, Kriyakova, and Kalla [13] gave solutions of fractional multi-order integral and differential equations using a Poisson-type transform during the same period. Again, Saxena, Mathai, and Haubold [268] obtained a solution of generalized fractional kinetic equations in a compact form containing Mittag-Leffler

function in 2002.During the same year,Podlubny [235] has given geometric and physical interpretations of Reimann-Liouville's left and right-sided fractional derivatives and integrals which are a milestone in the field of fractional calculus.

In 2003, Samko [259] came out with Hardy inequality in the generalized Lebesgue spaces. In 2004, Saxena, Mathai, and Haubold [270] applied fractional calculus in developing unified fractional kinetic equations. Again, Jain and Pathan [116] developed Weyl fractional integral operators in 2004. During the same year, Saxena, Mathai, and Haubold [270] investigated the solution of a unified form of fractional kinetic equation.Also, Yadav and Purohit [332] obtained certain transformations for the basic hypergeometric functions employing fractional q-derivative in the same period.

In 2006, Sharma and Jain [293] obtained a correlation between double Dirichlet average of $x^t \log x$ and fractional derivative. In the same year,Kiryakova [136] developed a solution of two Saigo's fractional integral operators in t he class of univalent functions. In 2007, Sharma and Jain [294] obtained a correlation between Dirichlet average of $\cosh x$ and fractional derivative. In the same year, Prieto,

Romero and Srivastava [238] present several key results for the generalized Lommel-Wright and related functions involving the Reimann-Liouville, the Weyl, and some other fractional calculus operators. In 2008,

Gafiychuk, Datsko, and Meleshko [79] applied fractional calculus in the Mathematical modeling of time-fractional reaction-diffusion systems. In the same year, Sharma [291] has given a new special function of fractional calculus namely M-series, and obtained its fractional integration and fractional differentiation. In 2009, Jumarie [120] developed a Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. Also, Sharma and Jain [297] have given a new special function of fractional calculus namely Generalized M-series in the same period.

In 2010, Chaurasia and Pandey [45] studied the computable extensions of generalized fractional kinetic equations in astrophysics. Kiryakova[139] presented the multi-index Mittag-Leffler functions as an important class of special functions of fractional calculus during the same year. Also, Soubhia, Camargo, Oliveira, and Vaz Jr. [304] obtained the application of fractional calculus in electrical engineering by developing a theorem for series in three-parameter Mittag-Leffler function, in the same period. In 2011, Bhalekar and Daftardar-goji [23] have studied a predictor-corrector scheme for solving nonlinear delay differential equations of fractional order. In the same year, Sharma and Jain [298] obtained the solution fra

ctional kinetic equation in astrophysics in terms of I-
function. Also, David, Linares, and Pallone [49] studied historical developm
ent and its applications infractional

order calculus. In 2012, Chaurasia [46] obtained the solution of the ti
me-space fractional diffusion equation by the integral transform method.

In 2013, Yang, Zhang and Long [334] introduced the Yang-
Fourier transformation for solving the heat-conduction in a semi-
infinite fractal bar. Gehlot [81] obtained the integral representation and certa
in properties of M-

series associated with fractional calculus, in the same year. Also, in th
e same year, Saxena, Ram, and Kumar [279] derived the solution of general
ized fractional kinetic equations by Sumudu transform.

In 2014, Khalil, Horani, Yousef, and Sababheh[347] have given a ne
w definition of fractional derivative. Also,

Caputo and Fabrizio[348] have developed a new definition of fraction
al derivative without singular kernel in 2015.

In 2016, Khalil, Horani, and Anderson[349] have given undetermined coeffi
cients for local fractional differential equations. Also, in the same year Gok
dogan, Unal, and Celik [350] have given existence and uniqueness theorems
for sequential linear conformable fractional differential equations.

In 2017, Chen and Katugampola [351] have developed Hermite–
Hadamard and Hermite–Hadamard–
Fejér type inequalities for generalized fractional integrals

Khan, Razzaq, and Ayaz[352] have developed some properties and applicati
ons of conformable fractional Laplace transform (CFLT) in 2018

In 2019, Hilfer and Luchko [353] have developed Desiderata for fractio nal derivatives and integrals.

In 2021, Baleanu and Agarwal[355] applied fractional calculus in thesky.

In 2022, Hashim, Sharadga, and Al-Refai[356] have given a reliable approach for solving delay fractional differe ntial equations.

Many prominent mathematicians and scientists namely A. M. Mathai, M. A. Pathan, R. K. Saxena, P. K. Banerjee, Pankaj Shrivastava, R. Y. Den is, S. N. Singh, S. P. Goel, Renu Jain, K. Nishimoto, P. Rusev, I. Dimovski, S.L. Kalla, V. Kiryakova, L. Boyadjiev, Anatoly A. Kilbas, Om Agrawal, J.A. Tenreiro Machado, Jocelyn Sabatier, Stefan Samko, Blas M. Vinagre. Dumitru Baleanu Juan J. Trujillo, Igor Podlubny, Ivo Petras, Tomas Skovra nek, Dagmar Bednarova, Andrea Mojzisova, and Yang Quan Chen contribut ed to this field by organizing international conferences, workshops, and sym posiums over the years.

An extraordinary contribution for developing and making familiar the Fractional Calculus and its Applications Virginia Kiryakovastarted and edit ed an international journal namely Journal of Fractional Calculus & Applied Analysis in 1998

The fractional calculus is related to special functions; therefore a brief discu ssion about the special functions is given in the next section.

1.3 A Brief Sketch of Special Functions:

Special functions are real or complex-valued functions of one or more real or complex variables which are specified so completely that their numerical values could be tabulated. The special functions were introduced in the seventeenth century when J. W. developed the theory of Gamma function long before Euler reached it. In the eighteenth century, the special functions were defined as solutions of differential equations emerging as mathematical models of certain problems in the sciences. The adjective 'special' of this

nomenclature can be ascribed to the simple fact that these functions allocated their origin to a special situation. Here we present a brief survey of the hypergeometric functions and their generalizations due to the key importance of hypergeometric functions in the study of special functions.

1.3.1 Hypergeometric Functions:-

The hypergeometric functions cover up many new areas of research in biology,chemistry,physics,
sciences, and engineering through mathematical modeling techniques. The Oxford Professor Wallis J. (1616-1703) in his work first used the term 'hypergeometric' to denote any series which was separate from the ordinary geometric series. During the next one hundred and fifty years, many notable mathematicians studied similar series, like Euler L. (1707-1783) who gave important results in this direction, amongst many other mathematicians and scientists.

In 1812, a famous German mathematician C. F. Gauss introduced the Gaussian hypergeometric series and presented it with F notation.

$$F(a,b;c;z) = 1 + \frac{a.b}{1.c}z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2 + \cdots,$$

where $(c \neq 0, -1, -2, \dots)$ (1.3.1.1)

Generalizations of this series are called multiple Gaussian hypergeom etric series.

To know Gaussian hypergeometric series, we require the following de finitions and symbols.

The Pochammer symbol $(a)_m$is given by

$$(a)_m = a(a+1)\dots(a+m-1), \quad \text{where } m = 1,\ 2,3,\dots$$

$$(a)_0 = 1,\ a \neq 0. \qquad (1.3.1.2)$$

Since $(1)_m = m!$, $(a)_m$ may be looked upon as a generalization of the elementary factori al; hence the symbol $(a)_m$ is also denoted as the factorial function.

Given the definition (1.3.1.1) we can show that

$$a_{2m} = 2^{2m}\left(\frac{a}{2}\right)_m\left(\frac{a}{2}+\frac{1}{2}\right)_m \text{ where } m = 1,\ 2,3,\dots \qquad (1.3.1.3)$$

From the above result (1.3.1.3) we can obtain the following formula k nown as Legendre's duplication formula for the Gamma function.

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}} \Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\dots\Gamma\left(z+\frac{m-1}{m}\right),$$

$$\text{where } m = 1,\ 2,3,\dots \qquad (1.3.1.4)$$

For every positive integer n, we have

$$(a)_{mn} = n^{mn} \prod_{j=1}^{n} \left(a + \frac{j-1}{n}\right)_m \text{, where } m = 1,\ 2\ ,3, \ldots \qquad (1.3.1.5)$$

which reduces to (1.3.1.3) when $n =$

2, Starting from (1.3.1.5) with$a = nz$. It can be proved that

$$\Gamma(nz) = (2\pi)^{\frac{1-n}{2}} n^{nz-\frac{1}{2}} \prod_{j=1}^{n} \left(z + \frac{j-1}{n}\right)_m,$$

where $z \neq 0, -\dfrac{1}{n}, \ -\dfrac{2}{n} \ldots; n = 1,2,3 \ldots$ (1.3.1.6)

This result is known as Gauss's multiplication theorem for the Gamm a function.

We can rewrite the definition (1.3.1.1) by the use of the Pochammer s ymbol defined in (1.3.1.2) as follows.

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \text{ where } c \neq 0, -1, -2, \ldots \qquad (1.3.1.7)$$

The infinite series in (1.3.1.7) obviously reduces to elementary geome tric series.

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots + z^n + \cdots \qquad (1.3.1.8)$$

in the following two special cases when

(i) $a = c$ and $b = 1$ and (ii) $a = 1$ and $b = c$. (1.3.1.9)

It is simply observed that the hypergeometric series in (1.3.1.7) converges absolutely within the unit circle $|z| < 1$, only if the denominator parameter c is neither zero nor a negative integer.

In case, $|z| < 1$ hypergeometric series (1.3.1.7), is absolutely convergent if $Re(c - a - b) > 0$.

If either or both of the numerator parameters a and b in (1.3.1.7) are zero or a negative integer, the hypergeometric series terminates and the question of convergence does not arise.

In fact, if $z = 1$in (1.3.1.7) we can obtain the well known Gauss's summation theorem,

$$ {}_2F_1(a,\ b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

where $Re(c-a-b) > 0,\ c \neq 0, -1,\ -2 \ldots$ (1.3.1.10)

A clear special case of (1.3.1.10) arise when the numerator parameter a or b is a non – integer, say$(-n)$, we have the summation formula

$$ {}_2F_1(-n,\ b;c;1) = \frac{(c-b)_n}{(c)_n}, $$

where $n = 0,\ 1,\ 2, \ldots\ c \neq 0, -1, -2$ (1.3.1.11)

which is equivalent to Vandermonde's convolution theorem

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} = \binom{\alpha+\beta}{n},\ n \geq 0, \qquad (1.3.1.12)$$

α and $\beta$are any complex numbers.

Several summation theorems for the hypergeometric series (1.3.1.7) when z takes on other special values are given in Bailey [21], Erdelyi et al. [67], and Slater [302].

In the Gaussian hypergeometric series $F\ (a,\ b,\ c;\ z),$ there are two numerator parameters a, b, and one denominator parameter c. A natural gen eralization of this series is obtained by introducing an arbitrary number of n umerator and denominator parameters. The resulting series

$${}_pF_q\left[\begin{matrix} a_1, \dots,\ a_p; \\ b_1, \dots,\ b_q; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!} \qquad (1.3.1.13)$$

is known as the generalized hypergeometric series. Here p and q are positive integers or zero (interpreting an empty product as 1) and the denom inator parameters $b_1, \dots,\ b_q$ take on complex values, provided that

$$b_q \neq 0,\ \ -1,\ \ -2,\ \ \dots; j = 1,\ 2,\ \ \dots q$$

(1.3.1.14)

Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restrictions (1.3.1.14), ${}_pF_q$series in (1.3.1.13).

(i) converges for $|z| < \infty\ if\ p \geq q,$

(ii) converges for $|z| < 1$ if $p = q + 1$ and

(iii) diverges for all z, $z \neq 0$ if $p > q + 1$.

Furthermore, if we set

$$w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \,, \tag{1.3.1.15}$$

then the series for ${}_pF_q$ with $p = q +$

1 is absolutely convergent for $|z| = 1$ if $Re(\mathrm{w}) > 0$

A significant special case of the series (1.3.1.13) is the Kummerian h ypergeometric series ${}_1F_1(a, c; z)$ in which case,

$p = q = 1$. Since

$$\lim_{|\alpha| \to \infty} \left\{ (a)_n \left(\frac{z}{a}\right)^n \right\} = \lim_{|\mu| \to \infty} \left\{ \frac{(\mu z)^n}{(\mu)_n} \right\} = z^n \tag{1.3.1.16}$$

for bounded z and $n = 0, 1, 2, \ldots$, we have,

$${}_1F_1(\, a, c; z) = \lim_{|b| \to \infty} {}_2F_1\left(\, a, b, c; \frac{z}{b}\right) \tag{1.3.1.17}$$

In consideration of the principle of confluence involved in

(1.3.1.17), Kummer's series ${}_1F_1(\, a, c; z)$ is also called the confluent hy pergeometric series.

An exciting generalization of the series ${}_pF_q$ is due to Fox and

Wright who studied the asymptotic expansion of the generalized hype rgeometric function defined by

$$ {}_p\psi_q\left[\begin{matrix}(a_1,A_1)\,\dots,\ (a_p,A_p);\\(b_1,B_1)\,\dots,\ (b_q,B_q);\end{matrix} z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+A_jn)}{\prod_{j=1}^{q}\Gamma(b_j+B_jn)}\frac{z^n}{n!} \quad (1.3.1.18) $$

where the coefficients$A_1,\dots,A_p$ and $B_1,\dots,B_q$ are positive real numbe rs such that

$$ 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geq 0, \quad \dots \quad (1.3.1.19) $$

With the help of (1.3.1.13), (1.3.1.18) and conditions given in (1.3.1.1 9) the following result can be obtained.

$$ {}_p\psi_q\left[\begin{matrix}(a_1,1)\,\dots,\ (a_p,1);\\(b_1,1)\,\dots,\ (b_q,1);\end{matrix} z\right]=\frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}\,{}_pF_q\left[\begin{matrix}a_1,\dots,\ a_p;\\b_1,\dots,\ b_q;\end{matrix} z\right] $$

$$ \dots \quad (1.3.1.20) $$

The massive success of the theory of hypergeometric series in one variable i nspired the development of corresponding theory in two and more variables. Appell [1888] was the first to introduce the theory of hypergeometric functi ons of two variables, The four Appell series were unified and generalized b y Kamp’e de F’eriet [1921] who defined a general hypergeometric series in two variables. A further generalization of the Kamp’e de F’eriet series is du e to Shrivastava and Daoust in [1969] who indeed defined an extension of ${}_p\psi_q$ series in two variables. On the other hand Lauricella [1893] further gen eralized the four Appell series $F_1,\ \dots,F_4$to the corresponding series in

n variables.

The hypergeometric series in one and more variables occur naturally in a large variety of problems in astronomy, statistics, physics, engineering, biological sciences,social sciences,and applied mathematics.

1.3.2 More Generalizations of Hypergeometric Functions:

The generalized hypergeometric function ${}_pF_q$ also has a variety of applications. The Barnes type contour integral representation of this function riveted many analysts to introduce successive generalizations. The attempts of MacRobert [168] and Meijer [190] obtained two special functions which are well known in the literature as the E-function and the G-function respectively.

1.3.2.1 The Mac Robert's E-Function:

The prodigious mathematician T. M. Mac Robert in the late 1930s attempted to give meaning to the symbol ${}_pF_q$ when $p > q + 1$, for the condition when $p \leq q + 1$, E-function is defined as follows:

$$E(p; a_r; q; b_s : x) = E(a_1, \ \ldots a_p; b_1, \ \ldots b_q : x)$$

$$= \frac{\Gamma(a_1) \ldots \Gamma(a_p)}{\Gamma(b_1) \ldots \Gamma(b_q)} {}_pF_q \left(a_1, \ \ldots a_p; b_1, \ \ldots b_q; -\frac{1}{x} \right) \quad \ldots \quad (1.3.2.1.1)$$

Where $x \neq 0$ if $p < q$ and $|x| > 1$ if $p = q + 1$;

while for $p \geq q + 1$, it can be put as

$$E(p;a_r;q;b_s:x)=\sum_{r=1}^{p}\frac{\prod_{s=1}^{p}{}'\,\Gamma(a_s-a_r)}{\prod_{t=1}^{q}\Gamma(b_t-a_r)}x^{\alpha r}$$

$$\times\ {}_{q+1}F_{p-1}\left[\begin{matrix}a_r-b_1+1,\ldots,\ a_r-b_q+1;\\ a_r-a_1,\ldots,{}^*a_r+a_p+1\ ;\end{matrix}(-1)^{p+q}x\right]\quad \ldots(1.3.2.1.2)$$

where $|x|<1$ if $p = q + 1$. The prime in Π represents the omission of the factor $\Gamma(a_r - a_r)$ the asterisk (*) in ${}_{q+1}F_{p-1}$ denotes the omission of the parameter $a_r - a_r + 1$ an empty product is to be interpreted as one and zero

or negative integer values of the a are tacitly excluded. The asymptotic expansion as $x \to \infty$,

$$-\frac{1}{2}(p-q+1)\pi<\arg z<\frac{1}{2}(p-q+1)\pi$$

of (1.3.2.1.2) is given by the right-hand side of (1.3.2.1.1).

1.3.2.2 The Meijer's G-function [190]:

The renowned mathematician C. S. Meijer in 1936 introduced the G-function which also provides an interpretation to the symbol ${}_qF_p$, when $p < q + 1$. This is in complete agreement with the one given by Mac Robert's E-function. In the subsequent definition, an empty product is interpreted as unity and $0 \le m \le q,\ 0 \le n \le p$. Meijer's G-function with the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ is defined as a Mellin-Barnes type integral as follows.

$$G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_1,\dots a_p\\ b_1,\dots b_q\end{matrix}\right.\right)\equiv G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\equiv G_{p,q}^{m,n}(z)\equiv G(z)$$

$$G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_1,\dots a_p\\ b_1,\dots b_q\end{matrix}\right.\right)=(2\pi i)^{-1}\int_L g(s)\,z^{-s}ds \quad \dots \ (1.3.2.2.1)$$

where $=\sqrt{-1}$, L is a suitable contour which will be discussed later on

,$z\neq 0$,

$$\mathrm{g}(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+s\right)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)} \quad \dots \ (1.3.2.2.2)$$

Here the parameters $a_1,\dots,a_p$ and $b_1,\dots,b_q$ are complex numbers such that no pole of $\Gamma\left(b_j+s\right), j=1,\dots,m$ coincides with any pole of

$\Gamma(1-a_k-s),\ k\ =\ 1,\dots,n$ that is,

$$-b_j-v\neq 1-a_k+\lambda,\ j=1,\dots,m;k=1,\dots,n;v,\ \lambda=0,1 \quad \dots \ (1.3.2.2.3)$$

This means that $a_k-b_j\neq v+\lambda,\forall j=1,\dots,m\ and\ k=1,\dots,n$. We also need that there is a strip in the complex s-plane that separates the poles of $\Gamma\left(b_j+s\right), j=1,\dots,m$, from those of $\Gamma\left(1-a_j-s\right),k=1,\dots,n$. The implication of this is shown in Fig. (1.1), [184].

Poles of $\Gamma\left(b_j+s\right)$	no poles	poles of $\Gamma\left(1-a_j-s\right)$
$j=1,\dots,m$	for $\mathrm{g}(s)$	$k=1,\dots,n$

*******   **********

****** c_1 c_2 *****

*******   ********

****

Figure (1.1)

An algebraic statement of Fig. (1.3.2.2.4) is that

$$min\{Re(b_j): j = 1, \dots, m\} < c_1 < \ Re(s)\} \ < c_2$$
$$< \ min\{Re(a_k): k = 1, \dots, n\} \qquad \dots \ (1.3.2.2.4)$$

where $Re(s)$ denotes the real part of s.

1.3.2.3 The Fox's H-Function [77]:

This function is an extension of the G-

function defined by Charles Fox [77] in 1961. The definition and the basic c onditions of existence for an H-function are as under:

$$H_{p,q}^{m,n}\left(z\left|\begin{matrix}a_p, \alpha_p\\ b_q, \beta_q\end{matrix}\right.\right) \equiv H_{p,q}^{m,n}(z) \equiv H(z)$$

$$H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_1, \alpha_1), \dots, (a_p, \alpha_p)\\ (b_1, \beta_1), \dots, (b_q, \beta_q)\end{matrix}\right.\right) = \frac{1}{(2\pi i)}\int_L h(s)\, z^{-s} ds \qquad \dots \ (1.3.2.3.1)$$

Where,

$$h(s) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j + \beta_j s\right) \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j s)} \qquad \ldots \; (1.3.2.3.2)$$

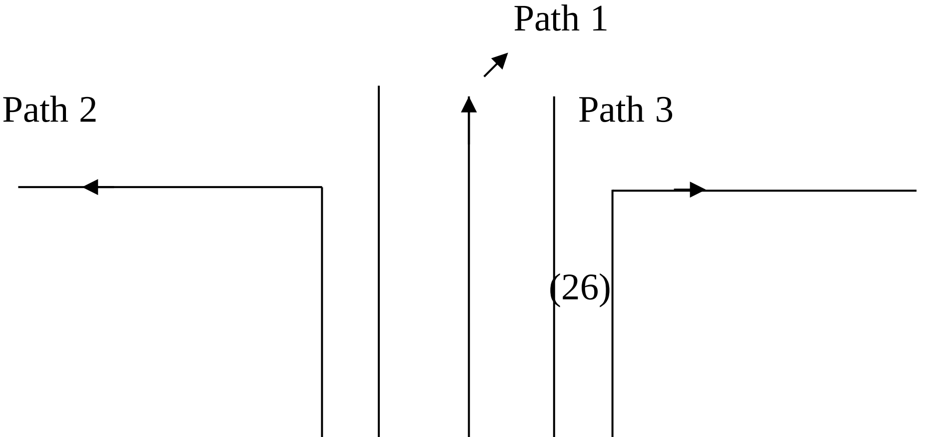
and L is a suitable path which will be described here. An empty product is interpreted as unity and it is assumed that the poles of $\Gamma\left(b_j + \beta_j s\right), j = 1, \ldots, m$, are separated from the poles of $\Gamma\left(1 - a_j - \alpha_j s\right),\ j = 1,\ \ldots, n.\ a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ are complex numbers, $\alpha_1, \ldots, \alpha_p,\ \beta_1, \ldots, \beta_q$ are positive real numbers. The poles of $\Gamma\left(b_j + \beta_j s\right), j = 1, \ldots, m$ are at the points

$$s = -\frac{b_j + v}{\beta_j},\ j = 1, \ldots m,\ v = 0,\ 1 \ldots$$

and the poles of $\Gamma\left(1 - a_j - \alpha_j s\right),\ j = 1,\ \ldots, n$ are at

$$s = \frac{(1 - a_k + \lambda)}{\alpha_k},\ k = 1, \ldots n,\ \lambda = 0,\ 1, \ldots$$

The condition of separability of these two sets of poles imposes that there be a strip in the complex s-plane where the H-function has no poles. There are three types of paths L possible. These correspond to paths 1, 2, 3, as are shown in Fig. (1.2), [49].



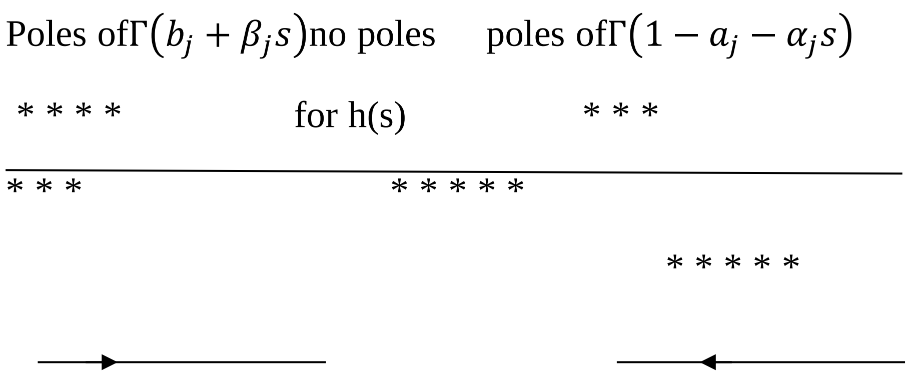


Figure (1.2)

For all practical problems where H-functions are to be applied we mainly require paths 2 and 3. It is to be pointed out that when more than one path L makes sense then it can be shown that they lead to the same function and hence there will be no ambiguity.

$$\text{let } \mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \text{ and } \beta = \left\{ \prod_{j=1}^{p} \alpha_j^{\alpha j} \right\} \left\{ \prod_{j=1}^{q} \beta_j^{-\beta j} \right\} \quad \ldots \; (1.3.2.3.4)$$

The H-function exists for the following cases [185]:

Case (i) $q \geq 1,\ \mu > 0,\ H(z)$ exists for all $z,\ z \neq 0$.

Case (ii) $q \geq 1,\ \mu = 0,\ H(z)$ exists for $|z| < \beta^{-1}$.

Case (iii) $p \geq 1,\ \mu > 0,\ H(z)$ exists for all $z,\ z \neq 0$.

Case (iv) $p \geq 1,\ \mu = 0,\ H(z)$ exists for $|z| > \beta^{-1}$.

In the above cases, it is assumed that the basic condition is satisfied that is the poles of $\Gamma(b_j + \beta_j s),\ j = 1, \ldots, m$ and $\Gamma(1 - a_j - \alpha_j s),\ j = 1,\ \ldots,\ n$ are separated. Note that in cases (1) and (ii) the H-function is evaluated as the sum of the residues at the poles of $\Gamma(b_j +$

$\beta_j s\big)$, $j = 1, \dots, m$ and in cases (iii) and (iv) the H-

function is evaluated as the sum of the poles of $\Gamma\big(1 - a_j - \alpha_j s\big)$, $j =$

$1, \; \dots, \; n$.

The H-

function is also generalized in 1982 when V.P. Saxena [284] discovered a n ew function in which the denominator parameters are in the summation for m of Gamma-

functions products, during the solution of dual integral equations involving H-function as Kernels. This is the so-called Saxena's I-function.

1.3.2.4 The Saxena's I-Function:

The I-function is defined as follows.

$$I[z] = I^{M,N}_{P_i,Q_i;R}\left[z \left| \begin{matrix} (a_j,\alpha_j)_{1,n}, \dots, (a_j,\alpha_j)_{n+1,P_i} \\ (b_j,\beta_j)_{1,m}, \dots, (b_j,\beta_j)_{m+1,Q_i} \end{matrix} \right.\right]$$

$$= \frac{1}{(2\pi i)} \int_L \phi(s)\, z^s ds \qquad \dots \quad (1.3.2.4.1)$$

Where $i = \sqrt{-1}, z \neq 0$ is a complex variable and

$$z^s = e^{[s(log|z|)+i \arg z]}$$

In which log $|z|$

denotes the natural logarithm and $arg\ z$ is not necessarily the principal valu e. An empty product is interpreted as unity. Also

$$\phi(s) = \frac{\prod_{j=1}^{M}\Gamma\left(b_j - \beta_j s\right)\prod_{j=1}^{N}\Gamma(1 - a_j - \alpha_j s)}{\sum_{i=1}^{R}\left[\prod_{j=M+1}^{Q_i}\Gamma(1 - b_{ji} - \beta_{ji} s)\prod_{j=N+1}^{P_i}\Gamma(a_{ji} - \alpha_{ji} s)\right]}$$

… (1.3.2.4.2)

where $P_i\ (i = 1, 2, \ldots, R)$, $Q_i\ (i =$ $1, 2, \ldots, R)$, M, N are integers satisfying $0 \leq N \leq P_i,\ 1 \leq M \leq$ $Q_i,\ (i =$ $1, 2, \ldots, R)$, R is finite, $\alpha_j,\ \beta_j,\ \alpha_{ji},\ \beta_{ji}$ are real and positive numbers and $a_j,\ a_{ji},\ b_j,\ b_{ji}$ are complex numbers such that none of the points

$$s = \frac{b_j + v}{\beta_j},\ j = 1, \ldots M,\ v = 0,\ 1,2, \ldots$$

which are the poles of $\Gamma\left(b_j - \beta_j s\right),\ (j = 1, 2, \ldots, M)$ and the points

$$s = \frac{(a_j - v - 1)}{a_j},\ j = 1, \ldots N,\ v = 0,\ 1,2, \ldots$$

which are the poles of $\Gamma\left(1 - a_j - \alpha_j s\right)$ coincide with one another i.e. $\alpha_j(b_h + v) \neq \beta_h(a_j - 1 - k)$ for $v,\ k = 0,\ 1,\ 2, \ldots h = 1,2, \ldots M, J = 1,2, \ldots R$; L is a contour which runs from, $\sigma - i\infty$ to $\sigma + i\infty$ (σ is real) in the complex s-plane such that the poles of $\Gamma\left(b_j - \beta_j s\right),\ (j = 1, 2, \ldots, M)$ lie to the right of L and the poles of $\Gamma\left(1 - a_j - \alpha_j s\right)(j = 1,2, \ldots, N)$ lie to the left of L respectively.

The contour integral (1.3.2.4.1) is absolutely convergent if either

$$A_i > 0,\ |\arg z| < \frac{1}{2} A_i \pi \quad \forall\ i \in \{1,\ 2,\ \dots R\}$$

Or

$$A_i \geq 0,\ |\arg z| < \frac{1}{2} A_i \pi,\quad Re(B+1) < 0,\quad \forall\ i \in \{1,\ 2,\ \dots R\}$$

… (1.3.2.4.3)

where $A_i = \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{P_i} \alpha_{ji} + \sum_{j=1}^{M} \beta_j - \sum_{j=N+1}^{Q_i} \beta_{ji} \quad \forall\ i \in \{1,\ 2,\ \dots R\}$

$$B = \frac{1}{2}(P_i - Q_i) + \sum_{j=1}^{Q_i} b_j - \sum_{j=1}^{P_i} a_i \quad \forall\ i \in \{1,\ 2,\ \dots R\} \qquad \dots \quad (1.3.2.4.4)$$

It is obvious from (1.3.2.4.1) that Saxena's I-function reduces to familiar Foxe's H-function when $R = 1$. Thus, a large class of special functions, including Bessel, Legendre, hypergeometric function, etc. turn out to be particular cases of I-functions.

There was no dead end of generalization in the field of special functions. Südland, Baumann, and Nonnenmacher [315] introduced the generalized form of I −function known as Aleph-function.

1.3.2.5. Südland, Baumann and Nonnenmacher's ℵ −(aleph) function:

This is the generalized form of I −function. The Aleph-function is given by Südland, Baumann and Nonnenmacher [315] (1998):

$$[z] = \aleph_{p_i,q_i,\tau_i;r}^{m,n}[z] = \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\dots,[\tau_i(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\dots,[\tau_i(b_j,\beta_j)]_{m+1,q_i}\end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)}\int_{\mathcal{L}} \Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)\, z^{-s} ds \qquad \dots \quad (1.3.2.5.1)$$

For all $z \neq 0$, where $\omega = \sqrt{-1}$ and

$$\Omega_{p_i,q_i,\tau_i;r}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} - B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji} + A_{ji}s)\right]}$$

... (1.3.2.5.2)

The integration path $\mathcal{L} = \mathcal{L}_{i\gamma\infty},\ \gamma \in R$extends from $\gamma - i\infty$ to $\gamma + i\infty$ and is such that the poles, assumed to be simple, of $\Gamma(1 - a_j - A_j s), j = 1, \dots n$ do not coincide with the poles of $\Gamma(b_j + B_j s),\ j = 1, \dots m$. the parameters p_i, q_i are non-negative integers satisfying $0 \le n \le p_i,\ 1 \le m \le q_i,\ \tau_i > 0$ for $i = 1, \dots r$. The parameters $A_j,\ B_j,\ A_{ji},\ B_{ji} > 0$ and $a_j, b_j,\ a_{ji}, b_{ji} \in C$. the empty

production in (1.3.2.5.2) is interpreted as unity. The existence conditions for the defining integral (1.3.2.5.1) are given below:

$$\varphi_\ell > 0, \quad |\arg z| < \frac{\pi}{2}\varphi_\ell \quad \forall\ \ell \in \{1,\ 2,\ \dots r\}$$

Or

$$\varphi_\ell \ge 0, \quad |\arg z| < \frac{\pi}{2}\varphi_\ell \quad Re\{(\zeta_\ell) + 1\} < 0, \quad \forall\ \ell \in \{1,\ 2,\ \dots R\}$$

… (1.3.2.5.3)

$$\text{where } \varphi_\ell = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_\ell \left(\sum_{j=n+1}^{p_\ell} A_{j\ell} - \sum_{j=m+1}^{q_\ell} B_{j\ell} \right)$$

$$\forall\ \ell \in \{1,\ 2,\ \ldots R\}$$

$$\zeta_\ell = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_\ell \left(\sum_{j=m+1}^{q_\ell} b_{j\ell} - \sum_{j=n+1}^{p_\ell} a_{j\ell} \right) + \frac{1}{2}(p_\ell - q_\ell)$$

$$\text{where,} \quad \ell = 1, \ldots r. \qquad \ldots \quad (1.3.2.5.4)$$

Further, we present an extension of an ordinary hypergeometric function wh ich is called basic analog or q-analog of Gauss hypergeometric function and obtained by addition of an ext ra parameter q. When q tends towards one, the basic hypergeometric functio n approaches a normal hypergeometric function. The basic hypergeometric f unctions have been observed significantly in providing an imminent into the structure of Ramanujan's identities and the Mock-Theta functions. Basic hypergeometric functions have found applications in various fields of sciences such as Lie

theory, elliptic functions, solid-state theory in physical chemistry, linear algebra, transient behaviors in elec trical cables, high energy particles physics, cosmology, number theory, and mechanical engineering, etc.

In the next section, we deal with the Mittag-Leffler function and its generalizations. Its importance is realized during the last two decades due to its direct contribution to the problems of astrophysi cs, biology, astronomy, applied sciences, social sciences, and engineering. H ille

and Tamarkin in 1920 have obtained a solution of Abel-Volterra type integral equation in terms of Mittag-Leffler function. Hence it has been observed that the Mittag-Leffler function occurs as the solution of fractional order differential equati ons (or fractional order integral equations).

1.4 Special Functions of Fractional Calculus:

The importance of special functions as a device of mathematical anal ysis is well known to scientists, mathematicians, social scientists, and engin eers dealing with the practical applications of differential equations. The sol ution of various problems from the heat conduction, electromagnetic waves, fluid mechanics, quantum mechanics, kinetic equations and diffusion equati ons, etc. lead obligatory to using the special functions. Special functions aris e as a solution of some basic ordinary differential equations and solving part ial differential equations usingthe separation of the variable method. The ver ity of the nature of the methods leading to special functions encouraged the increase of the number of special functions used in applications.

The other special functions (most of them being generalized Hyper-geometric ${}_pF_q$ -functions), such symbols were till newly less popular, unfamiliar, and still u nknown. There existed various integral and differential formulae for them b ut unfortunately quite unusual for each of the special functions and scattered in the literature without any mutual idea to relate them.

Special functions of mathematical physics can be represented as generalized Hyper-geometric ${}_pF_q$ –functions or more usually as Majer's G-functions. We have seen the manual books of the "classical calculus" era as the Bateman project; Luke [166], Abramowitz and Stegun [4], Mathai [184], Mathai and Saxena [185], etc. for their definitions and examples,

There is a growing interest and use of classes of special functions, ref erred to as "Special Function of Fractional Calculus" such as examples: the Mittag–Leffler functions, the Wright-Bessel (Bessel-Maitland) functions, the Wright generalized hypergeometric function ${}_p\psi_q$, t he Fox H-functions and increasing the number of their specifications, involving sets o f "fractional" multi-indices and closely related to operators and equations of the fractional mult i-order.

1.4.1 The Multi-Index Mittag-Leffler Function:

A class of special functions of Mittag-Leffler type that are multi-index analogs of $E_{\alpha,\beta}$ by replacing the indices $\alpha = 1/\rho\,,\ \beta = \mu$ by two sets of indices $(\alpha = 1/\rho_1\,, 1/\rho_2 \dots 1/\rho_m),\quad \beta = (\mu_{1,}\mu_2 \dots \mu_m)$For integer $m > 1$, let $\rho_1 \dots \rho_m > 0$ and $\mu_{1,}\mu_2 \dots \mu_m$ be arbitrary real (complex) numbers. Employing these "m ulti-indices" the multi-index Mittag-Lefflerfunctions (multi-MLF) are defined in the series:

$$E_{\frac{1}{\rho_i},\mu_i}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1)\Gamma(\mu_m + k/\rho_m)} \dots \quad (1.4.1.1)$$

The same kind of functions has been considered also by Luchko and Kiryakova [165] and the called those Mittag-Leffler functions of vector index.

1.4.2 The Generalized Multi-Index Mittag-Leffler Function:

The generalized multi-index Mittag-Leffler function was defined and studied by Saxena and Nishimoto in 2010.

$$E_{\rho,k}[(\alpha_i, \delta_i)_{1,m}; z] = \sum_{k=0}^{\infty} \frac{(\rho)_{kn} z^k}{\prod_{j=1}^{m} \Gamma(\alpha_i + \delta_i n)\ n!} \quad \dots \quad (1.4.2.1)$$

1.4.3 The Mittag-Leffler Function:

The Mittag-Leffler function acquainted with by Mittag-Leffler [197] in 1903 is defined as

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \ (\alpha \in C,\ Re(\alpha) > 0) \quad \dots \quad (1.4.3.1)$$

A generalization of the Mittag-Leffler function is given by Wiman [328] in 1905 defined as follows:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \ (\alpha, \beta \in C,\ Re(\alpha) > 0,\ Re(\beta) > 0)$$

... (1.4.3.2)

Prabhakar [237] familiarized a generalization of (1.4.3.2) in 1971 in t he form

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k x^k}{\Gamma(\alpha k + \beta)\ k!},$$

$$(\alpha, \beta, \gamma \in C,\ Re(\alpha) > 0),\ Re(\beta) > 0 \qquad \ldots \quad (1.4.3.3)$$

Where $(\gamma)_k$ is the Pochammer symbol.

It is an entire function with $\rho = [Re(v)]^{-1}$.

For $\gamma = 1$, this function coincides with (1.4.3.2), while for $\gamma = \beta = 1$ with (1.4.3.1) :

$$E_{\alpha,\beta}^{1}(x) = E_{\alpha,\beta}(x),\ E_{\alpha,1}^{1}(x) = E_{\alpha}(x) \qquad \ldots \quad (1.4.3.4)$$

We also have

$$\phi(\beta, \gamma; x) = {}_1F_1(\beta, \gamma; x) = \Gamma\gamma\ E_{1,\gamma}^{\beta}(x) \qquad \ldots \quad (1.4.3.5)$$

$$E_{\alpha,\beta}^{\gamma}(x) = \frac{1}{\Gamma\gamma} H_{1,2}^{1,1}\left[-x \left|\begin{matrix} (1-\gamma,\ 1) \\ (0,1),\ (1-\beta, \alpha) \end{matrix}\right.\right],\ Re(\alpha) > 0; \alpha, \beta, \gamma\ \in C$$

$$\ldots \quad (1.4.3.6)$$

For $\gamma = 1$(1.4.3.6) gives rise to the following result for the generalized Mittag-Leffler function.

$$E_{\alpha,\beta}(x) = H_{1,2}^{1,1}\left[-x \left|\begin{matrix} (0,\ 1) \\ (0,1),\ (1-\beta, \alpha) \end{matrix}\right.\right],\ Re(\alpha) > 0; \alpha, \beta \in C$$

… (1.4.3.7)

If we further take $\beta = 1$ in (1.4.3.7) we find that

$$E_{\alpha}(x) = H_{1,2}^{1,1}\left[-x\left|\begin{matrix}(0,\ 1)\\(0,1),\ (0,\ \alpha)\end{matrix}\right.\right],\ Re(\alpha) > 0; \alpha \in C \qquad \text{...} \quad (1.4.3.8)$$

1.4.4 The Agarwal's Function:

The Agarwal's Function is a generalization of the Mittag-Leffler function given by Agarwal (1953) as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{\left(k+\frac{\beta-1}{\alpha}\right)}}{\Gamma(\alpha k + \beta)} \qquad \text{...} \quad (1.4.4.1)$$

This function is exciting to the fractional-order system theory and its Laplace Transform, given by the Agarwal as

$$L\{E_{\alpha,\beta}(z^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha} - 1} \qquad \text{....} \quad (1.4.4.2)$$

This function is the $(\alpha - \beta)$ order fractional derivative of the function (Robotnov (1969) and Hartley (1998)), with argument $\alpha = 1$.

1.4.5 The Robotnov and Hartley's Function:

The following function was introduced (Hartley and Lorenzo, 1998) during solving of the fundamental linear fractional order differential equation :

$$F_q[-a,\ t] = t^{q-1} \sum_{n=0}^{\infty} \frac{(-a)^n t^{nq}}{\Gamma(nq+q)}, \quad q > 0 \qquad \text{... (1.4.5.1)}$$

This function had been studied by Robotnov (1969, 1980) concerning heredi tary integrals for application to solid mechanics. The

a significant property of this function is the power and simplicity of it s Laplace Transform

$$L\{F_q[a,\ t]\} = \frac{1}{s^q - a}, \quad q > 0 \qquad \text{... (1.4.5.2)}$$

1.4.6 The Miller and Ross Function:

Miller and Ross (1993, pp80 and 309-351) introduce another function as the basis of the solution of the fractional-order initial value problems. It is defined as

$$E_t(v, a) = \sum_{k=0}^{\infty} \frac{a^k t^{k+v}}{\Gamma(v+k+1)} \qquad \text{... (1.4.6.1)}$$

And its Laplace Transform

$$L\{E_t(v, a)\} = \frac{s^{-v}}{s-a}, \quad Re(v) > 1 \qquad \text{... (1.4.6.2)}$$

1.4.7 The Wright Function $W_{\alpha,\beta}(z)$:

The Wright function, denoted by $W_{\alpha,\beta}(z)$ is so named in honor of E. Maitland Wright [239], the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the o utline of the asymptotic theory of partitions.The function is defined by the s eries representation, convergent in the whole z-complex plane,

$$W_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k + \beta)}, \quad \alpha > -1, \qquad \beta \in C \qquad \ldots \quad (1.4.7.1)$$

So $W_{\alpha,\beta}(z)$ is an entire function.

And its Laplace transform, we have

$$L\{W\ (t;\ \alpha,\ \beta; s)\} = \sum_{k=o}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \frac{1}{s^{k+1}} \qquad \ldots \quad (1.4.7.2)$$

Or

$$L\{W\ (t;\ \alpha,\ \beta; s)\} = s^{-1} E_{\alpha,\ \beta}(s^{-1})C \qquad \ldots \quad (1.4.7.3)$$

This is Mittag-Leffler function.

1.4.8 The Mainardi Function:

The Mainardi function is a particular case of the Wright function. The Mainardi function and its applications are very useful to solve problems in various fields like physics, applied science, and engineering. The Mainardi f unction is defined as:

$$M\ (z, \alpha) = \sum_{k=o}^{\infty} \frac{(-1)^k}{k!} \frac{z^k}{\Gamma(-\alpha(k+1)+1)} \qquad \ldots \quad (1.4.8.1)$$

Where $\alpha, \in C$, $R(\alpha) > 0, z \in C$ and C is the set of a complex number.

1.4.9 The K_4 –Function:

This function is given by Kishan Sharma [289] and defined as

$$K_4^{(\alpha,\beta,\gamma),(a,c);(r;s)}(z) = \sum_{k=o}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{(\gamma)_k a^k (z-c)^{(k+\gamma)\alpha-\beta-1}}{\text{K}! \ \Gamma((k+\gamma)\alpha-\beta)}$$

... (1.4.9.1)

Here $v \in C$ and $(a_i)_k (i = 1,2,3, \dots p)$ and $(b_j)_k (j = 1,2,3, \dots q)$ are the Pochammar symbols.

1.4.10 The K_2 –Function:

This function is given by Kishan Sharma [290] and defined as

$$K_2(a_1 \ \dots a_p; , b_1 \dots b_q; z \ ) = \sum_{k=o}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{a^k z^{k+v}}{\Gamma(k+v+1)}$$

... (1.4.10.1)

Here $v \in C$ and $(a_i)_k (i = 1,2,3, \dots, p)$ and $(b_j)_k (j = 1,2,3, \dots, q)$ are the Pochammar symbols.

1.4.11 The M-series:

The M-series is a particular case of the H-function of Inayat Hussain, [110]. It plays a special role in the applications of fractional calculus operators and in the solutions of fractional order differ ential equations. The Hypergeometric function and Mittag-

Laffler function follow as its particular case. Therefore, it is very interesting . The M-series was introduced by Sharma [291]:

$$ {}_pM_q^{\alpha}\left(a_1\ .\ .\ .a_p;,b_1\ .\ .\ .\ b_q;z\ \right) $$

$$ =\sum_{k=o}^{\infty}\frac{(a_1)_k\ .\ .\ .\ \left(a_p\right)_k}{(b_1)_k\ .\ .\ .\left(b_q\right)_k}\frac{z^k}{\Gamma(\alpha k+1)} \quad ...\ (1.4.11.1) $$

Here, α,β ϵ C, $R(\alpha)>0,(a_j)_k(b_j)_k$are pochammer symbols.

1.4.12 The Generalized M-series:

The Generalized M-series is given by Sharma and Jain [] in 2009 and defined as follows:

$$ {}_pM_q^{\alpha,\beta}\left(a_1\ ...\,a_p;b_1\ ...\ b_q;z\right) $$

$$ =\sum_{k=o}^{\infty}\frac{(a_1)_k\ \ ...\ \left(a_p\right)_k}{(b_1)_k\,...\left(b_q\right)_k}\frac{z^k}{\Gamma(\alpha k+\beta)} \qquad ...\quad (1.4.12.1) $$

Here $\alpha,\ \beta\ \epsilon\ C$,$R(\alpha)>0,\ \ R(\beta)>0$; $(a_j)_k\ ,(b_j)_k$ are pochammer symbols.

1.4.13 The R-Function:

The R-function is introduced by Lorenzo and Hartly (1999).

$$ R_{q,v}[a,c,t]=\sum_{n=0}^{\infty}\frac{(a)^n(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)} \qquad ...\quad (1.4.13.1) $$

The Laplace Transform of the R-function is

$$L\{R_{q,v}[a,c,t]\} = \sum_{n=0}^{\infty} \frac{(a)^n}{\Gamma\big((n+1)q-v\big)} L((t-c)^{(n+1)q-1-v})$$

$$\dots \quad (1.4.13.2)$$

On taking $c = 0$, we have

$$L\{R_{q,v}[a,0,t]\} = \sum_{n=0}^{\infty} \frac{(a)^n}{\Gamma\big((n+1)q-v\big)} L\big((t)^{(n+1)q-1-v}\big) \dots (1.4.13.3)$$

From Erdelyi (1954), we have

$$L(t^v) = \Gamma(v+1)s^{-v-1},\ Re(v) > -1,\ Re(s) > 0. \quad \dots \ (1.4.14.4)$$

Applying the above equation, we get,

$$L\{R_{q,v}[a,0,t]\} = \sum_{n=0}^{\infty} \frac{(a)^n}{s^{(n+1)q-v}},\ Re\big((n+1)q-v\big) > 0,\ Re(s) > 0$$

$$L\{R_{q,v}[a,0,t]\} = \frac{1}{s^{-v}} \sum_{n=0}^{\infty} \frac{(a)^n}{s^{(n+1)q}},$$

$$Re\big((n+1)q-v\big) > 0,\ Re(s) > 0 \quad \dots \ (1.4.13.5)$$

This can be written as a geometric series that converges when $|a/s^q| < 1$. it can be shown by the long division that

$$L\{R_{q,v}[a,0,t]\} = \frac{s^v}{s^q - a},\ Re(q-v) > 0,\ Re(s) > 0 \quad \dots \ (1.4.13.6)$$

If $c \neq 0$ and using the shifting theorem then Laplace Transform of R-function

$$L\{R_{q,v}[a,c,t]\} = \frac{e^{-cs}s^{v}}{s^{q}-a}, \; c \geq 0, Re((n+1)q-v) > 0, \; Re(s) > 0$$

1.5. Definition and Properties of Fractional Integrals and Derivatives:

In this section, we present definitions and properties of various operat ors of fractional calculus. These include the Riemann-Liouville fractional integral and differential operator, Weyl operators and C aputo operator, etc.

1.5.1 The Right-Sided Riemann-Liouville Fractional Integral:

The right-sided Riemann-Liouville fractional integral of order α is defined by Miller and Ross [15, p. 45], Samko [260]:

$$ {}_{a}^{RL}D_{t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > a \qquad \ldots \quad (1.5.1.1)$$

where, $Re(\alpha) > 0$.

1.5.2 The Right-Sided Riemann-Liouville Fractional Derivative:

The right-sided Riemann-Liouville fractional derivative of order α is defined as

$$ {}_{a}^{RL}D_{t}^{\alpha} f(t) = \left(\frac{d}{dx}\right)^{n} \{I_{a}^{n-\alpha} f(t)\}, \; Re(\alpha) > 0, \; n = [Re(\alpha)+1]$$

… (1.5.2.1)

where $[\alpha]$ represents an integral part of the number α.

1.5.3 Riemann-Liouville Left-Sided Fractional Integrals:

The Riemann-Liouville left-sided fractional integrals of orderα.

Let $f(x) \in L(a,\ b),\ \alpha \in C,\ Re(\alpha) > 0$, then

$${}_aI_x^{\alpha} f(x) = {}_aD_x^{-\alpha} f(x) = \ I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}},\ x > a$$

… (1.5.3.1)

1.5.4 Riemann-Liouville Right-Sided Fractional Integrals:

TheRiemann-Liouville right-sided fractional integrals of order α

Let $f(x) \in L(a,\ b),\ \alpha \in C,\ Re(\alpha) > 0$, then

$${}_xI_b^{\alpha} f(x) = {}_xD_b^{-\alpha} f(x) = \ I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(x-t)^{1-\alpha}},\ x < b$$

... (1.5.4.1)

1.5.5 Riemann-Liouville Left-Sided Fractional Derivative:

The Riemann-Liouville left-sided fractional derivative of orderα

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \quad (n = [\alpha]+1) \ldots (1.5.5.1) $$

Where $[\alpha]$ denotes an integral part ofα.

1.5.6 Riemann-Liouville Right-Sided Fractional Derivative:

The Riemann-Liouville right-sided fractional derivative of orderα.

$$ {}_xD_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(\mathrm{n}-\alpha)}\left(\frac{d}{dx}\right)^n \int_x^b \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \quad (n = [\alpha]+1) \ldots (1.5.6.1) $$

where $[\alpha]$ denotes an integral part of α.

1.5.7 Modified Riemann-Liouville Fractional Derivative:

The Modified Riemann-Liouville fractional derivative of order α:

$$ {}_0D_x^\alpha = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\xi)^\alpha (f(\xi)-f(0))d\xi, \quad 0 < \alpha < 1) $$

… (1.5.7.1)

1.5.8 Caputo Fractional Derivative:

The Caputo fractional derivative of order α> 0 is introduced by Caput o [37] in the form (if $m-1 < \alpha \leq m,\ Re(\alpha) > 0,\ m \in N$):

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^m(\tau)d\tau}{(t-\tau)^{\alpha+1-m}} $$

$$= \frac{d^m f(t)}{dt^m}, \text{ if} \alpha = m \dots \quad (1.5.8.1)$$

where $\frac{d^m f(t)}{dt^m}$ is the n-th derivative of order m of the function $f(t)$for$t$.

Or

$${}_0^C D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f'(t)}{(x-t)^\alpha} dt, \text{ where } 0 < \alpha < 1) \quad \dots \quad (1.5.8.2)$$

1.5.9 The Weyl Fractional Integral:

$${}_x W_\infty^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty (x-t)^\alpha f(t) dt \ ,$$

$$\text{where } \alpha \in C, \ Re(\alpha) > 0, (-\infty < x < \infty) \quad \dots \quad (1.5.9.1)$$

1.5.10 The Weyl Fractional Derivative:

$${}_x D_\infty^\alpha f(x) = \ D_-^\alpha f(x) = (-1)^m \left(\frac{d}{dx}\right)^m {}_x W_\infty^{m-\alpha} f(x)$$

$$= (-1)^m \left(\frac{d}{dx}\right)^m \frac{1}{\Gamma(m-\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{1+\alpha-m}} dt \ , \ (-\infty < x < \infty)$$

where $\alpha \in C,\ m \in N, m-1 < \alpha < m$ ... (1.5.10.1)

$$I_{0-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} f(t) dt \qquad \text{... (1.5.11.2)}$$

are one-dimensional Riemann-
Liouville and Weyl integral operators, respectively.

1.6.1 Basic Properties of Fractional Integrals:

Fractional integrals have the following properties:

1. Fractional integrals obey the semi-
group property which is as follows:

$$_aI_x^{\alpha}\, {_aI_x^{\beta}} \emptyset = {_aI_x^{\alpha+\beta}} \emptyset = {_aI_x^{\beta}}\, {_aI_x^{\alpha}} \emptyset \qquad \text{... (1.6.1.1)}$$

$$_xI_b^{\alpha}\, {_xI_b^{\beta}} \emptyset = {_xI_b^{\alpha+\beta}} \emptyset = {_xI_b^{\beta}}\, {_xI_b^{\alpha}} \emptyset$$

2. The integration by parts for fractional integrals is defined by

$$\int_a^b f(x) ({_aI_x^{\alpha}} g) dx = \qquad \text{... (1.6.1.2)}$$

1.6.2 Basic Properties of Weyl Integral:

1. Weyl fractional integral obeys the semigroup properties, i.e.

$$_xw_{\infty}^{\alpha}\, {_xw_{\infty}^{\beta}} f = {_xw_{\infty}^{\alpha+\beta}} f = {_xw_{\infty}^{\beta}}\, {_xw_{\infty}^{\alpha}} f \qquad \text{... (1.6.2.1)}$$

2. Weyl fractional integral obeys the Parseval equality which is also known as fractional integral by parts:

$$\int_{0}^{\infty} f(x)({}_{x}w_{\infty}^{\alpha} g(x))dx = \int_{0}^{\infty} ({}_{x}w_{\infty}^{\alpha} f(x))g(x)dx \qquad \dots \quad (1.6.2.2)$$

In the present thesis, an attempt has been made to derive some theoretical applications of fractional calculus in the field of mechanical engineering, electrical engineering, and physics. We have introduced a fractional generalization of the standard kinetic equation and a new special function given by authors and also established the solution for the computational

extension of the Advanced fractional kinetic equation. Also, the 1-Dimensional fractal heat-conduction problem in a fractal semi-infinite bar has been developed by local fractional calculus (Calculus of arbitrary order) employing the analytical Advanced Yang-Fourier transforms method. Besides, we have obtained a solution of generalized Fractional integrodifferential equation of LCR circuit using hypergeometric series in terms of Mittag-Leffler function. In addition, we have obtained the closed-form solution of fractional differential equation associated with Newton's law of fractional order and fractional harmonic oscillator problem in terms of the Mittag-Leffler function.

Chapter 2

Fractional Calculus Approach in RLC circuit using Hypergeometric Series

2.1 Introduction:

The fractional calculus approach is applied in solving differenti al equation which is associated with an electrical circuit i.e. RLC circuit usi ng hypergeometric series. The solution of the fractional differential equation of the RLC circuit comes in the form of the Mittag-Leffler function and Ali et.al.[8] results are special cases of our main result.

2.2 Electrical Circuit[8]:

In this section, we present the three elements of the RLC electrical circuit w here C is a capacitance, L is inductance, R is resistance and we consider her e the only positive value of all these constants.

The constitutive equations associated with three elements of the RLC electri cal circuit are defined as under:

The voltage drop across resistance R=$U_R(t) = RI(t)$,

Where I (t) is current.

The voltage drop across inductor L= $U_L(t) = L\frac{dI}{dt}$

And the voltage drop across capacitance C= $U_c(t) = \frac{1}{c}\int_0^t I(v)dv$

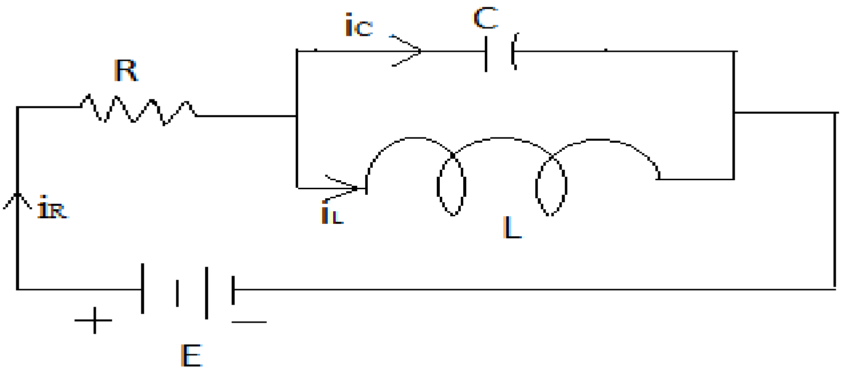
Kirchhoff law:The algebraic sum of the voltage drop around any closed circ uit is equal to the resultant EMF in the circuit.

By applying the Kirchhoff law in the non-homogeneous second-order ordinary differential equations. We get

$$\text{RC}\frac{d^2U_c(t)}{dt^2} + \frac{dU_c(t)}{dt} + \frac{R}{L}U_c(t) = \frac{d}{dt}\theta(t)$$

… (2.2.1)

Where $U_c(t)$ is the voltage on the capacitor, it is similar to the inductor as we can see in the figure because these are connected in parallel [8].



Three element LCR electrical circuit

Fig-1

Again, consider another non-homogeneous second-

order ordinary differential equation associated with current on the inductor a

s follows:

$$\text{RLC}\frac{d^2I_L(t)}{dt^2} + L\frac{dI_L(t)}{dt} + RI_L(t) = \theta(t)$$

… (2.2.2)

Using the constitutive equation for the inductor, these two non-

homogeneous second-

order ordinary differential equations can be led to correspondent integrodiff

erential equations, then we get

$$R\frac{di_c(t)}{dt} + \frac{1}{c}i_c(t) + \frac{R}{LC}\int_0^t i_c(v)dv = \frac{d}{dt}\theta(t)$$

… (2.2.3)

$$RC\frac{dU_L(t)}{dt} + U_L(t) + \frac{R}{L}\int_0^t U_L(v)dv = \theta(t)$$

… (.22.4)

We consider the initial condition $I_c(t) = 0$ at $t =$ 0 i.e. the initial current on the capacitor is zero and we get the solution in terms of an exponential function [8]

2.3. Fractional integrodifferential equation:

The fractional integrodifferential equation with current on the capacitor is as :

$$R\frac{d^\alpha}{dt^\alpha} i_c(t) + \frac{1}{c} i_c(t) + \frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}\, i_c(v)dv = \frac{d\theta(t)}{dt}$$

… (2.3.1)

The classical integrodifferential equation is associated with the RLC electrical circuit because for α=1 we improve the result get in equation (3.1). Its replacement is very important in discussing the corresponding numerical problem for a particular value of the parameter because the solution is obtained in terms of a closed expression [8]

The Laplace integral transform

$$L[i_c(t)] = F(s) = \int_0^\infty e^{-st}\, i_c(t)dt\ ,\quad Re(s) > 0$$

… (2.3.2)

Let $\theta(t)$= hypergeometricfunction in equation (3.2)

$$R\frac{d^\alpha}{dt^\alpha} i_c(t) + \frac{1}{c} i_c(t) + \frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}\, i_c(v)dv = \frac{d\theta(t)}{dt}$$

$$R\frac{d^\alpha}{dt^\alpha}i_c(t)+\frac{1}{c}i_c(t)+\frac{R}{LC}\frac{1}{\Gamma(\alpha)}$$

$$\int_0^t (t-v)^{\alpha-1}\, i_c(v)dv = \frac{d}{dt}\left[{}_pF_q(\mathrm{t})\right]$$

… (3.3)

$$R\frac{d^\alpha}{dt^\alpha}i_c(t)+\frac{1}{c}i_c(t)+\frac{R}{LC}\frac{1}{\Gamma(\alpha)}$$

$$\int_0^t (t-v)^{\alpha-1}\, i_c(v)dv = \frac{d}{dt}\left[\sum_{k\,=\,0}^{\infty}\frac{(a_1\ )_k \ldots.(\ a_p)_k}{(b_1\ )_k \ldots.(\ b_q)_k}\frac{t^k}{k!}\right]$$

… (2.3.3)

$$R\frac{d^\alpha}{dt^\alpha}i_c(t)+\frac{1}{c}i_c(t)+\frac{R}{LC}\frac{1}{\Gamma(\alpha)}$$

$$\int_0^t (t-v)^{\alpha-1}\, i_c(v)dv = \left[\sum_{k\,=\,0}^{\infty}\frac{(a_1\ )_k \ldots.(\ a_p)_k}{(b_1\ )_k \ldots.(\ b_q)_k}\frac{kt^{k-1}}{\Gamma(k+1)}\right]$$

… (2.3.4)

Applying the Laplace transform of both sides, we get

$$Rs^\alpha F(s)+\frac{F(s)}{C}+\frac{R}{LC}\frac{F(s)}{S^\alpha}=\sum_{k\,=\,0}^{\infty}\frac{(a_1\ )_k \ldots.(\ a_p)_k}{(b_1\ )_k \ldots.(\ b_q)_k}\frac{k\Gamma(k)}{\Gamma(k+1)}\frac{1}{s^k}$$

… (2.3.5)

$$F(s)[Rs^\alpha+\frac{1}{C}+\frac{R}{LC\ S^\alpha}]$$

$$= \sum_{k=0}^{\infty} \frac{(a_1\ )_k \dots.(\ a_p)_k}{(b_1\ )_k \dots.(\ b_q)_k} \frac{1}{s^k}$$

… (2.3.6)

$$RF(s) = \sum_{k=0}^{\infty} \frac{(a_1\ )_k \dots.(\ a_p)_k}{(b_1\ )_k \dots.(\ b_q)_k} \frac{1}{s^k} \frac{1}{[S^\alpha + \frac{1}{RC} + \frac{1}{LC\ S^\alpha}]}$$

… (2.3.7)

Let $a = \frac{1}{RC}$, $b = \frac{1}{LC}$

$$RF(s) = \sum_{k=0}^{\infty} \frac{(a_1\ )_k \dots.(\ a_p)_k}{(b_1\ )_k \dots.(\ b_q)_k} \frac{1}{s^k[S^\alpha + a + \frac{b}{S^\alpha}]}$$

… (2.3.8)

$$RF(s) = \sum_{k=0}^{\infty} \frac{(a_1\ )_k \dots.(\ a_p)_k}{(b_1\ )_k \dots.(\ b_q)_k} \frac{S^{\alpha-k}}{[S^{2\alpha} + aS^\alpha + b]}$$

… (2.3.9)

$$F(s) = \frac{1}{R} \sum_{k=0}^{\infty} \frac{(a_1\ )_k \dots.(\ a_p)_k}{(b_1\ )_k \dots.(\ b_q)_k} \frac{S^{\alpha-k}}{[S^{2\alpha} + aS^\alpha + b]}$$

… (2.3.10)

Taking the inverse Laplace transform of both sides, then we have

$$i_c(t) = \frac{1}{R} \sum_{k=0}^{\infty} \frac{(a_1\ )_k \dots.(\ a_p)_k}{(b_1\ )_k \dots.(\ b_q)_k} L^{-1} \left\{ \frac{S^{\alpha-k+1-1}}{S^{2\alpha} + AS^\alpha + B} \right\}$$

… (2.3.11)

We know the following relation by [12]

$$L^{-1}\left\{\frac{S^{\gamma-1}}{S^{\alpha}+AS^{\beta}+B}\right\}=t^{\alpha-\gamma}\sum_{r=0}^{\infty}(-A)^{r}$$

$$t^{(\alpha-\beta)r}E_{\alpha,\ \alpha+1-\gamma+(\alpha-\beta)r}^{r+1}(-Bt^{\alpha})$$

… (2.3.12)

Valid for $\left|\frac{AS^{\beta}}{S^{\alpha}+B}\right|<1,\ \alpha\geq\beta$

Using the relation (3.13), we get,

$$L^{-1}\left\{\frac{S^{\alpha-k+1-1}}{S^{2\alpha}+aS^{\alpha}+b}\right\}=t^{\alpha+k-1}\sum_{r=0}^{\infty}(-a)^{r}\ t^{\alpha r}E_{2\alpha,\ \alpha+k+\alpha r}^{r+1}(-bt^{2\alpha})$$

… (2.3.13)

Comparing the above these equations (3.13) and (3.14), then we get

$$i_{c}(t)=\frac{1}{R}\sum_{k\ =\ 0}^{\infty}\frac{(a_{1}\ )_{k}\ ....(\ a_{p})_{k}}{(b_{1}\ )_{k}\ ....(\ b_{q})_{k}}L^{-1}\left\{\frac{S^{\alpha-k+1-1}}{S^{2\alpha}+aS^{\alpha}+b}\right\}$$

… (2.3.14)

$$i_{c}(t)=\frac{1}{R}\sum_{k\ =\ 0}^{\infty}\frac{(a_{1})_{k}\ ...\left(a_{p}\right)_{k}}{(b_{1})_{k}\ ...\left(b_{q}\right)_{k}}$$

$$\left\{t^{\alpha+k-1}\sum_{r=0}^{\infty}(-a)^{r}\ t^{\alpha r}E_{2\alpha,\ \alpha+k+\alpha r}^{r+1}(-bt^{2\alpha})\right\}$$

... (2.3.15)

Here $E^{\rho}_{\mu,w}(t)$ is the Mittag-Leffler function of three parameters

Special Cases: 1. When $\theta(t) =$

${}_2F_1(\mathrm{a}_1, \mathrm{a}_2;\ \mathrm{b}_1; \mathrm{t})=\sum_{k\,=\,0}^{\infty} \frac{(a_1\ )_k(\ a_2)_k}{(b_1\ )_k} \frac{t^k}{k!}$is a Gauss's Hypergeometric function [9] then equation (3.16) reduces to

$$i_c(t) = \frac{1}{R} \sum_{k\,=\,0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k}$$

$$\left\{t^{\alpha+k-1} \sum_{r=0}^{\infty} (-a)^r\, t^{\alpha r} E^{r+1}_{2\alpha,\ \alpha+k+\alpha r}(-bt^{2\alpha})\right\} \quad .$$

…(2.3.16)

2. When $\theta(t) =$

${}_1F_1(\mathrm{a}_1;\ \mathrm{b}_1; \mathrm{t})=\sum_{k\,=\,0}^{\infty} \frac{(a_1)_k}{(b_1)_k} \frac{t^k}{k!}$is a confluenthypergeometric function[9]then e quation (3.17) reduces to

$$i_c(t) = \frac{1}{R} \sum_{k\,=\,0}^{\infty} \frac{(a_1)_k}{(b_1)_k} \left\{t^{\alpha+k-1} \sum_{r=0}^{\infty} (-a)^r\, t^{\alpha r} E^{r+1}_{2\alpha,\ \alpha+k+\alpha r}(-bt^{2\alpha})\right\}$$

…(2.3.17)

3. When we put $(a_1\ )_{k,} \dots, \left(a_p\right)_k = 1$ and $(b_1\ )_{k,} \dots, \left(b_q\right)_k =$ 1 and k=1 in equation (3.17) then we get Ali's [8] result

$$i_c(t) = \frac{1}{R}\left\{t^{\alpha}\sum_{r=0}^{\infty}(-a)^r\, t^{\alpha r} E_{2\alpha,\ \alpha+1+\alpha r}^{r+1}(-bt^{2\alpha})\right\}$$

…(2.3.18)

4. When we put $(a_1\ )_{k,} \dots, \left(a_p\right)_k = 1$ and $(b_1\ )_{k,} \dots, \left(b_q\right)_k = 1$ and k=2

in equation (3.17) then we get Ali's [8] result

$$i_c(t) = \frac{1}{R}\left\{t^{\alpha+1}\sum_{r=0}^{\infty}(-a)^r\, t^{\alpha r} E2_{2\alpha,\ \alpha+2+\alpha r}^{r+1}(-bt^{2\alpha})\right\}$$

…(2.3.19)

This completes the analysis.

2.4 Conclusion:

The applications of fractional calculus can be seen in many areas.It has been played an important role in electrical engineering. In this chapter,we have o btained the closed-form solution of fractional integrodifferential equation associated with RLC circuitusing the hypergeometric functions in terms of Mittag-Leffler function and Ali's [8] results are special cases of our result.

Chapter 3

FRACTIONAL KINETIC EQUATIONS NEW PARADIGM

3.1Introduction:

The present chapter aims to explore the behavior of physical and biological systems from the point of view of fractional calculus. Fractional calculus, integration, and differentiation of arbitrary or fractional order provide new tools that expand the descriptive power of calculus beyond the familiar integer-order concepts of rates of change and area under a curve. Fractional calculus adds new functional relationships and new functions to the familiar family of exponentials and sinusoids that arise in the area of ordinary linear differential equations. Among such functions that play an important role, we have the Euler Gamma function, the Euler Beta function, the Mittag-Leffler functions, the Wright and Fox functions, M-Function,K-functionThe first accurate use of a derivative of non-integer order is due to the French mathematician S. F. Lacroix in 1819 who expressed the derivative of non-integer order ½ in terms of Legendre's factorial symbol Γ.

We give the new special function, calledNew modifiedGeneralized $\mathcal{M}$ function [15], which is the most generalization ofthe $\mathcal{M}$ function.[14].

Here, we give first the notation and the definition of the New SpecialNew modifiedGeneralized $\mathcal{M}$function, introduced by the authors as follows:

$$
{}^{\alpha,\beta,\gamma,\delta,\rho,\tau}{}_{p}\mathcal{M}_{q}^{k_1,\dots k_p,\ \ l_{1,}\dots l_q;c}(t)
$$

$$
=\sum_{n=o}^{\infty}\frac{(a_1)_n\cdots(a_p)_n(\tau)_n(\gamma)_n}{(b_1)_n\cdots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{k_1^n\dots k_p^n}{l_1^n\dots l_q^n}\frac{\prod_{i=1}^{p}(m_i)\,(t-c)^{(n+\gamma)\alpha-\beta-1}}{\prod_{i=1}^{p}(n_i)!\,n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)}
$$

There are p upper parameters $a_{1,}\ a_{2,}\ \dots a_p$and q lower parameters

$b_{1,}\ \ b_{2,}\dots b_q$,$\alpha,\ \ \beta,\ \ \gamma,\delta,\rho,m\epsilon C$,$Re(\alpha)>0$,$Re(\beta)>0$,$Re(\gamma)>$

0,$Re(\delta)>0$, $Re(\rho)>0$, $Re(\tau)>0Re(\alpha\gamma-\beta)>$

$0$and $(a_j)_k(b_j)_k$are pochammer symbols and $k_1,\dots k_p,\ \ l_{1,}\dots l_q$are constants

.The function (1) is defined when none of the denominator parameters

$b_j s,\ \ j=$

$1,2,\dots q$ is a negative integer or zero. If any parameter a_j is negative then th

e function (1) terminates into a polynomial in $(t-c)$.

3.2 Relationship of the ${}^{\alpha,\beta,\gamma,\delta,\rho,\tau}{}_{p}\mathcal{M}_{q}^{k_1,\dots k_p,\ \ l_{1,}\dots l_q;c}$Function and Other Special Functions:

In this section, we defined the relationship ofNew modified Generaliz

ed $\mathcal{M}$functionand various special functions.

If we put $(\delta)_{nk}=(\delta)_n\quad and\ (\rho)_{nk}=(\rho)_n\ (\tau)_n=$

$1$and m_i=1 Then it converts into $\mathcal{M}\ function$ [15]

$$ {}^{\alpha,\beta,\gamma,\delta,\rho}{}_{p}\mathcal{M}_{q}^{k_1,\ldots k_p,\ \ l_1,\ldots l_q;c}(t). $$

$$ = \sum_{n=o}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \ldots k_p^n}{l_1^n \ldots l_q^n} \frac{(t-c)^{(n+\gamma)\alpha-\beta-1}}{\prod_{i=1}^{p}(n_i)!\, n!\, \Gamma\big((n+\gamma)\alpha-\beta\big)} $$

... (3.2.1)

(i) For$\prod_{i=1}^{p}(n_i)! = 1,\ (\tau)_n = 1$ Then Equation (i) Converts in $\mathcal{M}$ function [14]

$$ {}^{\alpha,\beta,\gamma,\delta,\rho}{}_{p}\mathcal{M}_{q}^{k_1,\ldots k_p,\ \ l_1,\ldots l_q;c}(t) = $$

$$ \sum_{n=o}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \ldots k_p^n}{l_1^n \ldots l_q^n} \frac{(t-c)^{(n+\gamma)\alpha-\beta-1}}{n!\, \Gamma\big((n+\gamma)\alpha-\beta\big)} $$

... (3.2.2)

(i) For $k_1 = a,\ k_2 \ldots k_p = 1,\ l_{1,} \ldots l_q = 1,\ \delta = 1$ and $\rho = 1, \prod_{i=1}^{p}(n_{i)}! \prod_{j=1}^{q}(n_j)! = n!\, (\tau)_n = 1 K_4$ −function is given by Sharma [13] (2012),

$$ {}^{\alpha,\beta,\gamma,1,1}{}_{p}\mathcal{M}_{q}^{a,\ \ 1;c}(t) = $$

$$ \sum_{n=o}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{a^n}{n!} \frac{(t-c)^{(n+\gamma)\alpha-\beta-1}}{\Gamma\big((n+\gamma)\alpha-\beta\big)} $$

... (3.2.3)

(ii) If we take no upper and lower parameter ($p = q = 0$) in equation (3) then the function reduces to the G Function, which was introduced by Lorenzo and Hartley [15] (1999).

$$ {}^{\alpha,\beta,\gamma,1,1}{}_{1}\mathcal{M}_{1}^{\alpha,\ \ 1;c}(t) = $$

$$ \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n (t-c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} = G_{\alpha,\beta,\gamma}(a,c,t) $$

... (3.2.4)

(iii) Taking $\gamma =$ 1, in equation (4), we get the R −function given by introduced by Lo renzo and Hartley [15] (1999).

$$ {}^{\alpha,\beta,1,1,1}{}_{1}\mathcal{M}_{1}^{\alpha,\ \ 1;c}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t-c)^{(n+1)\alpha-\beta-1}}{n!\Gamma\big((n+1)\alpha-\beta\big)} $$

$$ = R_{\alpha,\beta}[a,t] \alpha > 0,\ \beta > 0, (\alpha-\beta) > 0 $$

... (3.2.5)

Now, we take $c = 0$, in various standard functions.

(iv) For $c =$ 0, in equation (4),the Generalized $\mathcal{M}$ function reduces to New Gener alized Mittag-Leffler Function [12]

$$ {}^{\alpha,\beta,\gamma,1,1}{}_{1}\mathcal{M}_{1}^{\alpha,\ \ 1}(t) = t^{\alpha\gamma-\beta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n (t)^{\alpha n}}{n!\Gamma\big((n+\gamma)\alpha-\beta\big)} $$

$$ = t^{\alpha\gamma-\beta-1} E_{\alpha,\alpha\gamma-\beta}^{\gamma}[a t^{\alpha}] $$

... (3.2.6)

(v) We take $\gamma = 1$, in (6) obtained Generalized Mittag-Leffler Function [12], we get

$$ {}^{\alpha,\beta,1,1,1}{}_{1}\mathcal{M}_{1}^{\alpha,\ \ 1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{(n+1)\alpha-\beta-1}}{\Gamma\big((n+1)\alpha-\beta\big)} = t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}[a t^{\alpha}] $$

... (3.2.7)

(vi) Further $\beta = \alpha - 1$in(6), this Generalized $\mathcal{M}$ function converts into Mittag-Leffler Function [6,7], we have

$$ {}^{\alpha,\alpha-1,1,1,1}_{1}\mathcal{M}_{1}^{\alpha,\ \ 1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{n\alpha}}{\Gamma(n\alpha+1)} = E_\alpha[at^\alpha] $$

... (3.2.8)

(vii) When $a = 1,\ c = 0$ and $\beta = \alpha - \beta$ in (4) then the Generalized $\mathcal{M}$ the function treats as Agarwal's Function [1]

$$ {}^{\alpha,\alpha-\beta,1,1,1}_{1}\mathcal{M}_{1}^{1,\ \ 1}(t) = \sum_{n=0}^{\infty} \frac{(t)^{n\alpha+\beta-1}}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}[t^\alpha] $$

... (3.2.9)

(viii) Robotnov and Hartley Function [15] is obtained from $\mathcal{M}$ function by putting $\beta = 0,\ a = -a,\quad c = 0$in(5), we have

$$ {}^{\alpha,0,1,1,1}_{1}\mathcal{M}_{1}^{-a,\ \ 1}(t) = \sum_{n=0}^{\infty} \frac{(-a)^n (t)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} = F_\alpha[-a,t] $$

... (3.210)

(ix) On substituting $\alpha = 1,\ \beta = -\beta$ in(5), we get Miller and Ross Function [5].

$$ {}^{1,-\beta,1,1,1}_{1}\mathcal{M}_{1}^{a,\ \ 1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{n+\beta}}{\Gamma(n+\beta+1)} = E_t[\beta,\ a] $$

... (3.2.11)

(x) Let us consider $c = 0$ in equation(4), this function converts into

Wright Function [9]. We have,

$$ {}^{\alpha,\beta,\gamma,1,1}{}_{1}\mathcal{M}_{1}^{\alpha,\ 1}(t) = \frac{t^{\alpha\gamma-\beta-1}}{\Gamma\gamma}\, {}_{1}^{0}\psi_{1}^{\cdot}\left[\begin{matrix}(\gamma,\ 1)\\(\alpha\gamma-\beta),\alpha;at^{\alpha}\end{matrix}\right] \quad \dots\ (3.2.12) $$

Where ${}_{1}^{0}\psi_{1}^{\cdot}(t)$ is a special case of wright's generalized Hypergeometric func tion ${}_{p}^{0}\psi_{q}^{\cdot}(t)$.

Or

(xi) Thus we get H-Function [9] from the last case.

$$ {}^{\alpha,\beta,\gamma,1,1}{}_{1}\mathcal{M}_{1}^{\alpha,\ 1}(t) = \frac{t^{\alpha\delta-\beta-1}}{\Gamma\gamma} H_{1,2}^{1,1}\left[-at^{\alpha}\left|\begin{matrix}(1-\gamma,\ 1)\\(0,1)(1-\alpha\gamma+\beta),\alpha\end{matrix}\right.\right] $$

... (3.2.13)

The Laplace transform of(1), from Lorenzo & Hartley [15] (1999) with shif ting theorem (Wylie, p.281) we have

$$ L\left\{{}^{\alpha,\beta,\gamma,\delta,\rho}{}_{p}\mathcal{M}_{q}^{k_1,\dots k_p,\ \ l_{1,}\dots l_q;c}(t)\right\} = $$

$$ \frac{(a_1)_n\ \dots\ (a_p)_n}{(b_1)_n \dots (b_q)_n \prod_{i=1}^{p}(n_i)!}\frac{1}{l_1^n \dots l_q^n}\frac{s^{\beta}e^{-cs}}{\{s^{\alpha}+(k_1^n \dots k_p^n)\}^{\gamma}} $$

... (3.2.14)

3. Governing Fractional Kinetic Equation:

Let us define an arbitrary reaction that is dependent on time $N = N(t)$. It is possible to calculate the rate of change dN/dt to a balance betwe en the destruction rated and the production rate p of N, then

$$ \frac{dN}{dt} = -d + p. $$

The production or destruction at time t depends not only on N(t) but also on the previous history $N(t_1), \quad t_1 < \ t$, of the variable N.

This was represented by Haubold and Mathai [99] as follows:

$$\frac{dN}{dt} = \ -d\ (Nt\ )$$

$$+\ p(Nt), \qquad \ldots \quad (3.3.1)$$

where $N(t)$ denotes the function defined by

$$Nt(t_1)\ =\ N(t\ -t_1),\ t_1 >\ 0.$$

Haubold and Mathai [2] considered a special case of this equation when spa tial fluctuations inhomogeneities in quantity $N(t)$ are neglected. this is give n by the equation

$$\frac{dN_i}{dt} = -c_i N_i(t)$$

$$\ldots \quad (3.3.2)$$

where the initial conditions are $N_i\ (t\ =\ 0)\ =$ N_0, the number density of species i at time $t\ =$ 0; constant c_i>0, is called standard kinetic equation and $c_i >$ 0 is a constant.

The solution of the equation (15) is as follows:\

$N_i(t)\ =$

$$N_0 e^{-c_i t} \qquad \ldots\ (3.3.3)$$

Or

$$N(t) - N_0 = c\, {}_0D_t^{-1} N(t)$$

... (3.3.4)

As $D_t^{-1}$are the integral operator, Haubold, and Mathai [2] described the frac tional generalization of the standard kinetic equation (15) as

$$N(t) - N_0 = c^{v}\, {}_0D_t^{-v} N(t)$$

... (3.3.5)

Where D_t^{-v} is the Riemann-
Liouville fractional integral operator; Miller and Ross [5]) defined by

$${}_0D_t^{-v} N(t) = \frac{1}{\Gamma(v)} \int_0^t (t - u)^{v-1} f(u) du, \quad R(v) > 0, \quad \ldots \quad (3.3.6)$$

The solution of the fractional kinetic equation (18) is given by (see Haubold and Mathai [2])

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^{vk}}{\Gamma(vk+1)} (ct)^{vk} \quad \ldots \quad (3.3.7)$$

Also, Saxena, Mathai, and Haubold [12] studied the generalizations of the fr actional kinetic equation in terms of the Mittag-
Leffler functions which is the extension of the work of Haubold and Mathai [2].

In the present work, we studied the generalized fractional kinetic equation. The advanced generalized fractional kinetic equation and its solution, obtain ed in terms of the $\mathcal{M}$ −function,

4 Advanced Generalized Fractional Kinetic Equations:

In this section, we investigate the solution of the advanced generalize d fractional kinetic equation. The results are obtained in a compact form in t erms of New modified Generalized $\mathcal{M}$ function. The result is presented in t he form of a theorem as follows:

Theorem 1:

If $b \geq 0, \; c > 0, \; \alpha > 0, \; \beta > 0, \; \gamma > 0, \; \delta > 0, \; \rho > 0$ and $(\gamma\alpha - \beta) > 0$ than for the solution of the Advanced generalized fractional kinetic equation

$$N(t) - N_0 \, {}_p\mathcal{M}_q^{\alpha,\beta,\gamma,\delta\tau,\rho} {}^{-c^{\alpha}, \; b_{1,}\dots b_n;b}(t) = - \sum_{r=1}^{n} \binom{n}{r} c^{r\alpha} D_t^{-r\alpha} \, N(t) \qquad \dots (3.4.1)$$

Then

$$N(t) = N_0 \, {}_p\mathcal{M}_q^{\alpha,\beta,(\gamma+n),\delta\tau,\rho} {}^{-c^{\alpha}, \; b_{1,}\dots b_n;b}(t)$$

$$\dots (3.4.2)$$

Proof. We have,

$$N(t) - N_0 \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n \ (\gamma)_n (\tau)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{(-c^\alpha)^n}{b_1^n \dots b_q^n} \frac{\prod_{i=1}^{p}(m_i) \ (t-b)^{(n+\gamma)\alpha-\beta-1}}{\prod_{i=1}^{p}(n_i)! \ n! \ \Gamma\big((n+\gamma)\alpha-\beta\big)}$$

$$= \ - \ \sum_{r=1}^{n} \binom{n}{r} c^{r\alpha} D_t^{-r\alpha} \ N(t)$$

$$\dots (3.4.3)$$

Taking the Laplace transforms of both the sides of equation (24), we get

$$L\{N(t)\} - L \ \left\{ N_0 \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n \ (\gamma)_n (\tau)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{(-c^\alpha)^n}{b_1^n \dots b_q^n} \frac{\prod_{i=1}^{p}(m_i)(t-b)^{(n+\gamma)\alpha-\beta-1}}{\prod_{i=1}^{p}(n_i)! n! \Gamma\big((n+\gamma)\alpha-\beta\big)} \right\} = \ =$$

$$-L\left\{ \sum_{r=1}^{n} \binom{n}{r} c^{r\alpha} \ {}_0D_t^{-r\alpha} \ N(t) \right\}$$

$$. \qquad \dots \ (3.4.4)$$

From Lorenzo & Hartley [160] (1999) using the shifting theorem for Laplac e transform, we have

$$N(s) - N_0 \frac{(a_1)_n \ \dots \ (a_p)_n \ (\tau)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{\prod_{i=1}^{p}(m_i)}{\prod_{i=1}^{p}(n_i)! \ n! \ b_1^n \ \dots b_q^n} \frac{s^\beta e^{-bs}}{(s^\alpha + c^\alpha)^\gamma}$$

$$= -\left\{ \sum_{r=1}^{n} \binom{n}{r} c^{r\alpha} s^{-r\alpha} \ N(s) \right\}$$

$$\dots (3.4.5)$$

Or,

$$N(s) - N_0 \frac{(a_1)_n \ \dots \ (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{(\tau)_n (\delta)_n}{(\rho)_n} \frac{\prod_{i=1}^{p}(m_i)}{b_1^n \dots b_q^n \prod_{i=1}^{p}(n_i)! \ n!} \frac{s^\beta e^{-bs}}{(s^\alpha + c^\alpha)^\gamma}$$

$$= -[{}^n_0c_1 c^{\alpha} s^{-\alpha} + {}^n_0c_2 c^{2\alpha} s^{-2\alpha} \ldots {}^n_0c_n c^{n\alpha} s^{-n\alpha}] N(s)$$

... (3.4.6)

$$N(s)(1 + c^{\alpha} s^{-\alpha})^n =$$

$$N_0 \frac{(a_1)_n \ \ldots \ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\tau)_n (\delta)_n}{(\rho)_n} \frac{\prod_{i=1}^{p}(m_i)}{b_1^n \ldots b_q^n \prod_{i=1}^{p}(n_i)!\, n!} \frac{s^{\beta-\alpha\gamma} e^{-bs}}{(1 + c^{\alpha} s^{-\alpha})^{\gamma}}$$

... (3.4.7)

$$N(s) = N_0 \frac{(a_1)_n \ \ldots \ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\tau)_n (\delta)_n}{(\rho)_n}$$

$$\frac{\prod_{i=1}^{p}(m_i)}{b_1^n \ldots b_q^n \prod_{i=1}^{p}(n_i)!\, n!} \frac{s^{\beta-\alpha\gamma} e^{-bs}}{(1 + c^{\alpha} s^{-\alpha})^{\gamma+n}}$$

... (3.4.8)

$$N(s) = N_0 \frac{(a_1)_n \ \ldots \ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\tau)_n (\delta)_n}{(\rho)_n} \frac{\prod_{i=1}^{p}(m_i)}{b_1^n \ldots b_q^n \prod_{i=1}^{p}(n_i)!} \frac{s^{\beta-\alpha(\gamma+n)+n\alpha} e^{-bs}}{(1 + c^{\alpha} s^{-\alpha})^{\gamma+n}}$$

... (3.4.9)

$$N(s) = N_0 \frac{(a_1)_n \ \ldots \ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\tau)_n (\delta)_n}{(\rho)_n} \frac{\prod_{i=1}^{p}(m_i)}{\prod_{i=1}^{p}(n_i)!\, b_1^n \ldots b_q^n} \sum_{n=o}^{\infty} \left(\frac{-c^{\alpha}}{s^{\alpha}}\right)^n$$

$$\frac{(\gamma+n)_n s^{\beta-\alpha\gamma} e^{-bs}}{n!}$$

... (3.4.10)

Now, taking inverse Laplace transform, we get $N(t) =$

$$N_0 \frac{(a_1)_n \ \ldots \ (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(\tau)_n (\delta)_n}{(\rho)_n \prod_{i=1}^{p}(n_i)!} \frac{1 \prod_{i=1}^{p}(m_i)}{b_1^n \ldots b_q^n}$$

$$\sum_{n=o}^{\infty} (-c^{\alpha})^n \frac{(\gamma+n)_n}{n!} L^{-1}\left\{s^{\beta-\alpha\gamma-\alpha n} e^{-bs}\right\}$$

$$\dots (3.4.11)$$

$$N(t) = N_0 \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{(\tau)_n (\delta)_n \prod_{i=1}^{p}(m_i)}{(\rho)_n \prod_{i=1}^{p}(n_i)!} \frac{(-c^\alpha)^n}{b_1^n \dots b_q^n} \frac{(\gamma+n)_n}{n!}$$

$$\frac{(t-b)^{\alpha\gamma+\alpha n-\beta-1}}{\Gamma((\gamma+n)\alpha-\beta)}$$

$$\dots (3.4.12)$$

$$N(t) = N_0 {}^{\alpha,\beta,(\gamma+n),\delta,\rho}{}_p\mathcal{M}_q^{-c^\alpha,\ \ b_1,\dots b_n;b}(t)$$

$$\dots (3.4.13)$$

This is the complete proof of the theorem.

5. Special Cases:

Corollary:1. If we take$(a_1)_n \ \dots \ (a_p)_n$ $, k = 1 = (b_1)_n \dots (b_q)_n,$

$(\tau)_n = 1\delta = 1,\ \rho = 1$and $b_1^n \dots b_q^n = 1 \prod_{i=1}^{p}(n_i)! = 1$and

$\prod_{i=1}^{p}(m_i) = 1$than for the solution of the Advanced generalized fractional kinetic equation

$$N(t) \text{–}\ N_0 {}^{\alpha,\beta,\gamma,1,1}{}_1\mathcal{M}_1^{-c^\alpha,\ \ 1;b}(t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\alpha} D_t^{-r\alpha}\ N(t)(35)$$

… (3.5.1)

There holds the result

$$N(t) = \ N_0 {}^{\alpha,\beta,(\gamma+n),1,1}{}_1\mathcal{M}_1^{-c^\alpha,\ \ 1;b}(t)$$

$$\dots (3.5.2)$$

Because of the relation (33), this result coincides with the main result of Chaurasia and Pandey [16].

Corollary: 2. If we put $b = 0$ in corollary (1) then the solution of the

Advanced generalized fractional kinetic equation reduces to the the special case of theorem (1) in Chaurasia and Pandey [16] (2010), given as follows:For the solution of

$$N(t) - N_0 {}^{\alpha,\beta,\gamma,1,1}{}_{1}\mathcal{M}_1^{-c^{\alpha},\ 1;0}(t) = -\sum_{r=1}^{n}\binom{n}{r} c^{r\alpha} D_t^{-r\alpha} N(t) \quad \dots (3.5.3)$$

There holds the result

$$N(t) = N_0 {}^{\alpha,\beta,(\gamma+n),1,1}{}_{1}\mathcal{M}_1^{-c^{\alpha},\ 1;0}(t) \quad \dots (3.5.4)$$

Corollary: 3. If we put $\beta = \gamma\alpha - \beta$ in corollary (1) then the solution

of the Advanced generalized fractional kinetic equation reduces to the

a special case of theorem (1) in Chaurasia and Pandey [16] (2010), which is given as follows:

For the solution of

$$N(t) - N_0 {}^{\alpha,\gamma\alpha-\beta,\gamma,1,1}{}_1\mathcal{M}_1^{-c^{\alpha},\ 1;b}(t)$$

$$= - \sum_{r=1}^{n} \binom{n}{r} c^{r\alpha} D_t^{-r\alpha} N(t)$$

... (3.5.5)

There holds the formula

$$N(t) = N_0 {}^{\alpha,\gamma\alpha-\beta,(\gamma+n),1,1}{}_1\mathcal{M}_1^{-c^{\alpha},\ 1;b}(t)$$

... (3.5.6)

Corollary: 4. If we put $b = 0$ in corollary (3) then the solution of the Advanced generalized fractional kinetic equation reduces to another special case of theorem (1) in Chaurasia and Pandey [16], which is given as follows:

For the solution of

$$N(t) - N_0 {}^{\alpha,\gamma\alpha-\beta,\gamma,1,1}{}_1\mathcal{M}_1^{-c^{\alpha},\ 1;0}(t) = - \sum_{r=1}^{n} \binom{n}{r} c^{r\alpha} D_t^{-r\alpha} N(t)$$

... (3.5.7)

There holds the formula

$$N(t) = N_0 {}^{\alpha,\gamma\alpha-\beta,(\gamma+n),1,1}{}_1\mathcal{M}_1^{-c^{\alpha},\ 1;0}(t)$$

... (3.5.8)

This completes the analysis.

6. Result and Discussion:

In this present endeavor, we have introduced a frac tional generalization of the standard kinetic equation and a new special func tion given by authors and also established the solution for the computational extension of the Advanced fractional kinetic equation. The results of the co mputational extension Advanced generalized fractional kinetic equation and its special cease are the same as the results of Chaurasia and Panday [17] (2 010).

Chapter 4

Advanced Yang-Fourier Transforms to Heat-Conduction in a Semi-Infinite Fractal Bar

Introduction:

The main aim of the present chapter to solve the 1-Dimensional fractal heat-conduction problem in a fractal semi-infinite bar has been developed by local fractional calculus (Calculus of arbi trary order) employing the analytical**Advanced**Yang-Fourier transformsmethod.

AdvancedYang-Fourier transforms which is obtained by the author by a generalization ofYang-Fourier transformsis a technique of fractional calculus for solving mathemat ical, physical, and engineering problems. The fractional calculus is continuo usly growing in the last five decades [1-7]. Most ofthe fractional ordinary differential equations have exact analytic solutions, while others required either analyticalapproximations or numerica l techniques to be applied, among them: fractional Fourierand Laplace transf orm [8,41], the heat-balance integral method [9-11], variation iteration method(VIM) [12-14], decomposition method [15,41], homotopy perturbation method [16,41], etc.

The problems in fractal media can be successfully solved by lo cal fractional calculus theory with problems for non-differential functions [25-

32]. Local fractional differential equations have been applied to model com plex systems of fractal physical phenomena [30-41] local fractional Fourier series method [38], Yang-Fourier transform [39, 40,41]

2. A New Special Function and AdvancedYang-Fourier transform and properties of Advanced Young -Fourier transform:

Here, we define a new special function Sas follows:

$$S = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k a^k}{(b_1)_k \ldots (b_q)_k \mathrm{k}!\ \Gamma(1+\alpha)} z^k, \ \ \alpha \in c, R(\alpha) > 0.$$

If we put a = 1 in the above function, then the S function converts int o the

M-series [42].

And If we put $\frac{(a_1)_k \ldots (a_p)_k a^k}{(b_1)_k \ldots (b_q)_k}$ = 1 in S- function, then the S- function converts into the Mittag-Leffler function [43].

Let us Consider$f(x)$ is local fractional continuous in$(-\infty,\ \infty)$ we denote as $f(x) \in Ca\ (-\infty,\ \infty)$ [32, 33, 35].

Let $f(x) \in Ca\ (-\infty,\ \infty)$The AdvancedYang-Fourier transform developed by authors written in the form [30, 31, 39, 40, 41]:

$$F_\alpha\{f(x)\} = f_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^{\infty} S_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x)(dx)^\alpha \quad \ldots (4.2.1)$$

When we put$\frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k \text{k!}} a^k$ =1, then it converts into the Yang-Fourier transform [41].

Then, the local fractional integration is given by [30-32, 35-37, 41]:

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b f(t)\,(dx)^\alpha = \frac{1}{\Gamma(\alpha+1)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)\ (\Delta t_j)^\alpha$$

... (4.2.2)

Where$\Delta t_j = t_{j+1} - t_j$, $\Delta t = \ max\{\Delta t_1,\ \Delta t_2,\ \Delta t_j, \ldots\}\{t_j,\ t_{j+1}\}$,$j = 0, \ldots,$ $N - 1$, t_0 = a, $t_N = b$, is a partitionof the interval $[a,\ b]$.

If $F_\alpha\ \{f(x)\} =$ $f_\omega^{F,\alpha}(\omega)$, then its inversion formula takes the form [30, 31, 39, 40,41]

$$f(x) = F_\alpha^{-1}\left[f_\omega^{F,\alpha}\ (\omega)\right] =$$

$$\frac{1}{\Gamma(\alpha+1)} \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} S_\alpha\ (-i^\alpha \omega^\alpha x^\alpha) f_\omega^{F,\alpha}\ (\omega)(d\omega)^\alpha$$

... (4.2.3)

When we put$\frac{(a_1)_k \cdots (a_p)_k a^k}{(b_1)_k \cdots (b_q)_k \text{k!}}$ =1, it converts into the Yang Inverse Fourier transform [41].

Some properties are shown as follows [30, 31]:

Let $F_\alpha\ \{f(x)\} = f_\omega^{F,\alpha}(\omega)$, and $F_\alpha\{g(x)\} =$ $f_\omega^{F,\alpha}(\omega),$ and let be two constants. Then wehave:

$$F_\alpha\{cf(x) + dg(x)\} cF_\alpha\{f(x)\} + dF_\alpha\{g(x)\}$$

... (4.2.4)

If $\lim_{|x|\to\infty} f(x) = 0$, then we have:

$$F_\alpha\{f^\alpha(x)\} = i^\alpha \omega^\alpha F_\alpha\{f(x)\}$$

... (4.2.5)

In eq. (5) the local fractional derivative is defined as:

$$f^\alpha(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha[f(x) - f(x_0)]}{(x - x_0)^\alpha}$$

... (4.2.6)

Where $\Delta^\alpha[f(x) - f(x_0)] \cong \Gamma(1+\alpha)\Delta[f(x) - f(x_0)]$

As a direct result, repeating this process, when:

$$f(0) = f^\alpha(0) = \cdots = f^{(k-1)\alpha,}(0) = 0$$

... (4.2.7)

$$F_\alpha\{f^{k\alpha}(x)\} = i^\alpha \omega^\alpha F_\alpha\{f(x)\}$$

... (4.2.8)

3. Heat conduction in a fractal semi-infinite

If a fractal body is subjected to a boundary perturbation, then the heat diffus es in-
depth modeled by a constitutive relation where the rate of fractal heat flux $\overline{q}(x,\ y,\ z,\ t)$ is proportional to the local fractional gradient of the temperat ure [32,41], namely:

$$\overline{q}(x,\ y,\ z,\ t) = -K^{2\alpha}\nabla^\alpha T(x,\ y,\ z,\ t)$$

... (4.2.9)

Here the pre-

factor $K^{2a}$is the thermal conductivity of the fractal material. Therefore, the fr actal heat conduction equation without heat generation was suggested in [32

] as:

$$K^{2\alpha}\frac{d^{2\alpha}T(x,\ y,\ z,\ t)}{dx^{2\alpha}} - \rho_\alpha c_\alpha \frac{d^{2\alpha}T(x,\ y,\ z,\ t)}{dx^{2\alpha}} = 0$$

... (4.2.10)

Where$\rho_\alpha$and $c_\alpha$are the density and the specific heat of the material, respecti vely.

The fractal heat-

conduction equation with a volumetric heat generation $g(x,\ y,\ z,\ t)$ can be described as [32,41]:

$$K^{2\alpha}\nabla^{2\alpha}T(x,\ y,\ z,\ t) + g(x,\ y,\ z,\ t)\rho_\alpha c_\alpha \frac{\partial^\alpha T(x,\ y,\ z,\ t)}{\partial t^\alpha}$$

... (4.2.11)

The 1-Dimensional fractal heat-conduction equation [32,41] reads as:

$$K^{2\alpha}\frac{\partial^{2\alpha}T(x,\ t)}{\partial x^{2\alpha}} - \rho_\alpha c_\alpha \frac{\partial^\alpha T(x,\ t)}{\partial t^\alpha} = 0, \quad 0 < x < \infty,\ t > 0$$

... (4.2.12a)

with initial and boundary conditions are:

$$\frac{\partial^\alpha T(0,\ t)}{\partial t^\alpha} = S_\alpha t^\alpha,\ T(0,\ t) = 0$$

... (4.2.12b)

The dimensionless forms of (12a, b) are [35, 41]:

$$\frac{\partial^{2\alpha}T(x,\ t)}{\partial x^{2\alpha}} = \frac{\partial^\alpha T(x,\ t)}{\partial x^\alpha} = 0$$

... (4.2.13a)

$$\frac{\partial^{\alpha}T(0,\ t)}{\partial x^{\alpha}} = S_{\alpha}t^{\alpha},\ T(0,\ t) = 0$$

$$\dots(4.2.13b)$$

Based on eq. (12a), the local fractional model for 1-D fractal heat-conduction in a fractal semi-infinite bar with a source term $g(x,\ t)$ is:

$$K^{2\alpha}\frac{\partial^{2\alpha}T(x,\ t)}{\partial x^{2\alpha}} - \rho_{\alpha}c_{\alpha}\frac{\partial^{\alpha}T(x,\ t)}{\partial t^{\alpha}} = g(x,\ t), \quad -\infty < x < \infty,\ t > 0$$

$$\dots(4.2.14a)$$

With

$$T(x,\ 0) = f(x),\ \ -\infty < x < \infty,$$

$$\dots(4.2.\ 14b)$$

The dimensionless form of the model (14a, b) is:

$$\frac{\partial^{2\alpha}T(x,\ t)}{\partial x^{2\alpha}} = \frac{\partial^{\alpha}T(x,\ t)}{\partial t^{\alpha}} = 0, \quad -\infty < x < \infty,\ t > 0$$

$$\dots(4.2.15a)$$

$$T(x,\ 0) = f(x),\ \ -\infty < x < \infty,$$

$$\dots(4.2.15b)$$

4. Solutions by the Generalized New Yang-Fourier transform method:

Let us consider that $F_{\alpha}\{T(x,t)\} = T_{\omega}^{F,\alpha}(\omega,t)$is the Advanced Yang-Fourier transform of $T(x,\ t)$, regarded as a non-differentiable function of x. Applying the Yang-Fourier transform to the first term of Eq. (15a), we obtain:

$$F_\alpha\left\{\frac{\partial^{2\alpha}T(x,\ t)}{\partial x^{2\alpha}}\right\} = (i^{2\alpha}\omega^{2\alpha})T_\omega^{F,\alpha}(\omega,t) = \omega^{2\alpha}T_\omega^{F,\alpha}(\omega,t)$$

$$\dots(4.1.16a)$$

On the other hand, by changing the order of the local fractional differentiati on and integration in the second term of eq.(15a), we get:

$$F_\alpha\left\{\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}T(x,\ t)\right\} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}T_\omega^{F,\alpha}(\omega,t)$$

$$\dots(4.2.16b)$$

For the initial value condition, the Yang-Fourier transform provides:

$$F_\alpha\{T(x,\ 0)\} = T_\omega^{F,\alpha}(\omega,0) = F_\alpha\{f(x)\} = f_\omega^{F,\alpha}(\omega)$$

$$\dots(4.3.16c)$$

Thus we get from eqn.(16a,b,c):

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}T_\omega^{F,\alpha}(\omega,t) + \omega^{2\alpha}T_\omega^{F,\alpha}(\omega,t) = 0,\ T_\omega^{F,\alpha}(\omega,0) = f_\omega^{F,\alpha}(\omega)$$

$$\dots(4.4.17)$$

This is an initial value problem of a local fractional differential equation wit h *t* as an independent variable and *w*as a parameter.

$$T(\omega,t) = f_\omega^{F,\alpha}(\omega)S_\alpha(-\omega^{2\alpha}t^{\alpha})$$

$$\dots(4.5.18a)$$

Hence, using the inversion formula, eqn. (3), we get:

$$T(x,t)=\frac{1}{(2\pi)^{\alpha}}\int_{-\infty}^{\infty}S_{\alpha,}(i^{\alpha}\omega^{\alpha}x^{\alpha})f_{\omega}^{F,\alpha}(\omega)S_{\alpha,}(-\omega^{2\alpha}t^{\alpha})(d\omega)^{\alpha}=(Mf)(x)$$

$$\dots(4.6.18b)$$

$$M_{\omega}^{F,\alpha}(\omega)=\frac{1}{(2\pi)^{\alpha}}S_{\alpha,}(-\omega^{2\alpha}t^{\alpha})$$

$$\dots(4.7.18c)$$

From [30, 32] we obtained,

$$F_{\alpha}\left\{S_{\alpha}\left(-\frac{\omega^{2\alpha}}{C^{2\alpha}}\right)\right\}=\frac{C^{\alpha}\pi^{\frac{\alpha}{2}}}{1.}\frac{1}{\Gamma(\alpha+1)}S_{\alpha,}\left(-\frac{C^{2\alpha}\omega^{2\alpha}}{4^{\alpha}}\right)$$

$$\dots(4.8.19a)$$

Let $C^{2\alpha}/4^{\alpha}=t^{\alpha}$. Then we get:

$$F_{\alpha}\left\{S_{\alpha}\left(-\frac{\omega^{2\alpha}}{4^{\alpha}t^{\alpha}}\right)\right\}=\frac{1}{\Gamma(\alpha+1)}\frac{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}}{.}+S_{\alpha}(-\omega^{2\alpha}t^{\alpha})$$

$$=\frac{1}{\Gamma(\alpha+1)}\frac{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}}{.}(2\pi)^{\alpha}M_{\omega}^{F,\alpha}(\omega)$$

$$\dots(4.9.19b)$$

Thus,$M_{\omega}^{F,\alpha}(\omega)$ have the inverse:

$$\frac{1}{(2\pi)^{\alpha}}\int_{-\infty}^{\infty}S_{\alpha}(i^{\alpha}\omega^{\alpha}x^{\alpha})M_{\omega}^{F,\alpha,}(\omega)(d\omega)^{\alpha}\;=$$

$$\frac{1}{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}}\frac{1}{(2\pi)^{\alpha}}\Gamma(\alpha+1)\;S_{\alpha}\left(-\frac{\omega^{2\alpha}}{4^{\alpha}t^{\alpha}}\right)$$

$$\dots(4.10.19c)$$

Hence, we get:

$$T(x,\ t) = (Mf)(x) =$$

$$\frac{\Gamma(1+\alpha)}{4^{\alpha} t^{\frac{\alpha}{2}} \pi^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} f(\xi) S_{\alpha}\left(-\frac{(x-\xi)^{2\ \alpha}}{4^{\alpha} t^{\alpha}}\right) (d\xi)^{\alpha}$$

... (4.11.20)

This completes the analysis.

Special case:

When we put $\frac{(a_1)_k \cdots (a_p)_k a^k}{(b_1)_k \cdots (b_q)_k \mathrm{k!}}$ =1 then the S-

function converts into the Mittag-

Leffler functionand solution of Advanced Yang Fourier Transforms convert

into Yang Fourier Transforms results [41]

$$T(x,\ t) = (Mf)(x)$$

$$= \frac{\Gamma(1+\alpha)}{4^{\alpha} t^{\frac{\alpha}{2}} \pi^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} f(\xi) E_{\alpha}\left(-\frac{(x-\xi)^{2\ \alpha}}{4^{\alpha} t^{\alpha}}\right) (d\xi)^{\alpha}$$

... (4.12.21)

Conclusions:

In this chapter, wepresented an analytical solution of 1-

Dimensional heat conduction in the fractal semi-

infinite bar by the Advanced Yang-Fourier transform of non-

differentiable functions. The above findings are very useful in solving the pr actical problems because we have applied a partial fractional differential eq uation on a Cantor set

Chapter 5

A Generalization of Truncated S-Fractional Derivative and Applications to Fractional Differential Equations

5.1 Introduction:

In this chapter, we aim to study the work of Ilhan and OnurKiymaz (2020) regarding the generalization of truncated M-Fractional derivative and applications to Fractional differential equationswhich based on the generalization of the truncated M-fractional derivative which was recently introduced [Sousa and de Oliveira, A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties, Inter. of Jour. Analy. and Appl., 16 (1), 83–96, 2018]. To do that, we used generalized S-series, which has a more general form than generalized M-series,Mittag-Leffler, and hypergeometric functions. We called this generalization a truncated S-series fractional derivative. This new derivative generalizes several fractional derivatives and satisfies important properties of the integer-order derivatives. Finally, we obtain the analytical solutions of some S-series fractional differential equationsFractional calculus is a field that is frequently studied by mathematicians because of its many applications used to model problems. In some recent studies, it is seen that mathematical models obtained by using various fractional derivatives have better overlapping wit h experimental data rather than the models with derivatives of integer order. Fractional derivative definitions may be used for different types of problem s. This situation led scientists to identify more general fractional operators. Especially in the last six years, several generalizations of some well-known fractional derivative operators have been addressed by many authors (see, for example [2, 3, 5, 6, 11, 18, 19, 33]). In addition to these studies, dif ferent fractional derivative operators having many features provided by the i nteger-order derivative operator were also studied (see [16, 17, 27–31] and the references therein). In 2014, Khalil et al. [17] introduced a new type of fractional derivative for f : [0,∞) → R, t > 0 and α ∈ (0,1)as

$$T_{\alpha}f(t)=\lim_{\in\to 0}\frac{f\ (t+\varepsilon t^{1-\alpha})-f\ (t)}{\varepsilon}$$

…(5.1.1)

They called it conformable fractional derivative. In the same ye ar, Katugampola [16] introduced the alternative and truncated alternative fra ctional derivatives for f : [0,∞) → R as

$$D^{\alpha}(f)(t)=\lim_{\in\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f\ (t)}{\varepsilon},\ \ t>0,\quad \alpha\epsilon(0,1)$$

… (5.1.2)

$$\text{And}D^{\alpha}{}_{i}(f)(t)=\lim_{\in\to 0}\frac{f\left(te_{i}{}^{\varepsilon t^{-\alpha}}\right)-f\ (t)}{\varepsilon},t>0,\ \ \alpha\epsilon(0,1)$$

... (5.1.3)

respectively. Here $e^{x}{}_{i} = \sum_{k=0}^{i} \frac{x^k}{k\,!}$ is the truncated exponential function.Re cently, Sousa and de Oliveira [27, 29] introduced the M-fractional and truncated M-fractional derivatives for f : [0,∞) → R as

$$D_{M,}^{\alpha:\beta} f\ (t) = \lim_{\in \to 0} \frac{f\left(tE_{\beta}\ (\varepsilon t^{-\alpha})\right) - f\ (t)}{\varepsilon}, \beta, t > 0, \alpha\epsilon(0,1)$$

…(5.1.4)

and $${}_{i}D_{M\,,}{}^{\alpha:\beta} f\ (t) = \lim_{\in \to 0} \frac{f\left(t_i E_{\beta}(\varepsilon t^{-\alpha})\right) - f\ (t)}{\varepsilon}, \quad \beta, t > 0, \alpha\epsilon(0,1)$$

...(5.1.5)

respectively, using one parameter Mittag-Leffler function [12]$E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}$, $\Re\ (\beta) > 0, z \in$ $c$and its truncated version.All the derivatives given above satisfy some prop erties of classical calculus, e.g.linearity, product rule, quotient rule, function composition rule, and chain rule. Besides, for all the operators given above the α-order derivative of a function is a multiple of $t^{1-\alpha} \frac{df}{dt}$. In 2021, Dhaneliya a ndSharma[37] defined a new special function S as follows:

$$S = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k a^k}{(b_1)_k \cdots (b_q)_k k!\ \Gamma(1+\alpha)} z^k\ ,\ \alpha \in c, R(\alpha) > 0.$$

Here, we define a new generalized special function namely Generalized S-function, which is defined as :

$${}_{p}S_{q\,,}{}^{\alpha:\beta,\gamma}(z) := {}_{p}S_{q\,,}{}^{\alpha:\beta,\gamma}\begin{bmatrix} a_1, \dots,\ a_p; \\ c_1, \dots,\ c_q; \end{bmatrix} = $$

$$\sum_{k\,=\,0}^{\infty} \frac{(a_1\ )_k \ldots.(\ a_p)_k}{(c_1\ )_k \ldots.(\ c_q)_k} \frac{a^k z^k}{K!\ \ \Gamma(\beta k + \gamma)}$$

If we put k=1 and $\gamma = 1$, then the above function converts into S-
function [37]

If we put a = 1and K! =
1, in the above function, then the Generalized S function converts into the Generalized M-
series [42]. Where β, γ, z ∈ C, p, q ∈ N, ℜ(β) > 0, c_i ≠0,−1,−2,...(i = 1,2,.,q) and and a =1,2,…Here, $(\alpha)_k$ is the Pochhammer symbol [1] which given b y $(\alpha)_v = \frac{\Gamma(\alpha+v)}{\Gamma(\alpha)}$, $\alpha, v \epsilon C$with the assume $(\alpha)_0$= 1. Note that if a_j(j=1,2 ,.,p) equals to z ero or a negative integer, then the series reduces to a polynomial. And If we put $\frac{(a_1)_k \ldots (a_p)_k a^k}{(b_1)_k \ldots (b_q)_k k!}$ = 1 in

S- function, then the S- function converts into the Mittag-Leffler function

[43]Generalized S-
series is convergent for all z if p ≤ q; it is convergent for |z| < δ = α^{α} if p = q + 1; and divergent if p > q + 1. When p = q + 1 and |z| = δ, the series can converge on conditions depending on the parameters. For more information about S-
series, we refer to [37] and the referencestherein.A generalization of truncat ed S-
fractional derivative

Most of the famous special functions can b e described as the special cases of the generalized

Generalized S-series:

$$
{}_1S_{1,}{}^{1:1,1}(1;1;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z,\ {}_1S_{1,}{}^{1:\beta,1}(1;1;z)
$$

$$
=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}=\ E_\beta(z),\ {}_1S_{1,}{}^{1:\beta,\gamma}(1;1;z)
$$

$$
=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+\gamma)}=\ E_{\beta,\gamma}(z),\ {}_1S_{1,}{}^{1:\beta,\gamma}(\sigma;1;z)
$$

$$
=\sum_{k=0}^{\infty}\frac{(\sigma)_k z^k}{\Gamma(\beta k+\gamma)}=E_{\beta,\gamma}{}^{\sigma}(z)\ ,\ {}_1S_{1,}{}^{1:\beta,\gamma}(a;c;z)
$$

$$
=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!}=\ \phi(a;c;z),\ {}_2S_{1,}{}^{1:1,1}(a;b;c;z)
$$

$$
=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}={}_2F_1(a,b;c;z),\ {}_pS_{q,}{}^{1:1,1}(z)
$$

$$
=\sum_{k=0}^{\infty}\frac{(a_1\ )_k\ \ldots.(\ a_p)_k}{(c_1\ )_k\ \ldots.(\ c_q)_k}\frac{z^k}{k!}={}_pF_q\begin{bmatrix}a_1,\ldots,\ a_p\\ c_1,\ldots,\ c_q\end{bmatrix};z\Big]\ ,
$$

Here, $E_{\beta,}\ E_{\beta,\gamma}$, $E_{\beta,\gamma}{}^{\sigma}$ are the one [23], two [32] and three para meters [24] Mittag-
Leffler functions; and also $\Phi, {}_2F_1$, ${}_pF_q$ are the confluent, Gauss, and general ized hypergeometric functions [1], respectively. Motivated by the above stu dies and the frequent use of Generalized M-
series in fractional operator theory (see [8–10,14, 21]), with the help of S-series, we first define a more general fractional derivative (truncated S-series fractional derivative) and investigate its properties like linearity, prod uct rule, the chain rule, etc. Then we extend some of the classical results in calculus like Rolle's theorem, mean value theorem, etc. We also introduce t he S-
series fractional integral and finally, we obtain the analytical solutions of or dinary and partial S-series fractional linear differential equations.**2**

Truncated S-series Fractional Derivative :

We first present the definitions of the truncated S-series and truncated S-series fractional derivative operator

Definition 1.

The truncated S-Series is defined for

β > 0 as $_{i}S_{p,q}^{a:\beta,\alpha}(t) = {}_{i}S_{p,q}^{a:\beta,\alpha}(t)\begin{bmatrix} a_1, \dots, & a_p \\ c_1, \dots, & c_q \end{bmatrix}; t\Big]$:

$$= \sum_{k=0}^{i} \frac{(a_1)_k \dots (a_p)_k}{(c_1)_k \dots (c_q)_k K!} \frac{a^k t^k}{\Gamma(\beta k+\gamma)}$$

… (5.2.1)

where β, γ,t∈ R, p,q∈ N, a_n, c_m∈ R, $c_m \neq 0$,−1,−2,...

(n = 1,2,..., p; m = 1,2,...,q) and a =1,2,…

Definition 2.

Let f : [0,∞) → R. For β > 0, t > 0 and α ∈ (0,1), the truncated S-series fractional derivative of order α of a function f is

$$_{i}D_S^{\alpha} f(t) = {}_{i}D_S^{\alpha}\begin{bmatrix} a_1, \dots, & a_p \\ c_1, \dots, & c_q \end{bmatrix} f(t) = \lim_{\epsilon \to 0} \frac{f[\Gamma(\gamma)t]\,{}_{i}S_{p,q}^{a:\beta,\alpha}(\varepsilon t^{-\alpha}) - f(t)}{\varepsilon}$$

…. (5.2.2)

where β, γ,t∈ R, p,q∈ N, a_n, c_m∈ R, $c_m \neq$ 0 ,−1,−2,...(n = 1,2,.., p; m = 1,2,..,q) and a=1,2,…and $_{i}S_{p,q}^{a:\beta,\alpha}$ is the truncated S-series given with (6)? If a truncated S-series fractional derivative of a function f exists then we called the function f is S-differentiable

Note that, if f is S-differentiable in some interval (0,a), a > 0 and $\lim_{t\to 0^+} {}_iD_S^{\alpha} f(0)$exists, then we define ${}_iD_S^{\alpha} f(0) =$ $\lim_{t\to 0^+} {}_iD_S^{\alpha} f(t)$Because Sousa and de Oliveira showed in [29] that, truncated M-fractional derivative is the generalization of the fractional derivative operators (1)-(4), it is enough to choose γ= p = q = 1 and $a_1 = c_1$, a=1and k=1in (7) for proving that all the fractional derivative operators (1)-(5) given above are the special cases of our definition. For the sake of shortness, throughout the paper, we assume that α,β, γ ∈ R, p, q ∈ N, β > 0, p > 0, q > 0, a_n, c_m∈ R and $c_m \neq$ 0,−1,−2,...(n = 1,2,..., p; m = 1,2,...,q). Also, we use the notation K instead of the constant$\frac{a_1 \cdots a_p}{c_1 \cdots c_q} \frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$ Now we begin our investigation with an important theorem.

Theorem 1.

If a function f : [0,∞) → R is S-differentiable at t_0> 0 for α ∈ (0,1], then f is continuous at t_0.Proof. Consider the identity

$$f[\Gamma(\gamma)t_0]\,{}_iS_{p,q}^{a:\beta,\alpha}(\varepsilon t^{-\alpha}) - f\ (t_0) = \frac{f[\Gamma(\gamma)t]\,{}_iS_{p,q}^{a:\beta,\alpha}(\varepsilon t^{-\alpha}) - f\ (t)}{\varepsilon}\epsilon$$

Applying the limit for ε → 0 on both sides, we get

$$\lim_{\varepsilon\to 0}[\Gamma(\gamma)t_0]\,{}_iS_{p,q}^{a:\beta,\alpha}(\varepsilon t^{-\alpha}) - f\ (t_0)$$

$$= \lim_{\varepsilon \to 0} \left(\frac{f[\Gamma(\gamma)t]_i S_{p,q}^{a:\beta,\alpha}(\varepsilon t^{-\alpha}) - f\ (t)}{\varepsilon} \right) \lim_{\varepsilon \to 0} \varepsilon$$

$$= \ {}_iD_S^{\alpha}\ f\ (t)\ \lim_{\varepsilon \to 0} \varepsilon = \ 0$$

Then, f is continuous at t_0. Besides, using the definition of the truncated S-series, we can write

$$f\left(\Gamma(\gamma)t\,{}_iS_{p,q}^{a:\beta,\alpha}(\varepsilon t^{-\alpha})\right) = f\left(\Gamma(\gamma)t\ \sum_{n\ =\ 0}^{i} \frac{(a_1\ )_k \ldots .(\ a_p)_k}{(c_1\ )_k \ldots .(\ c_q)_k} \frac{a^k(\varepsilon t^{-\alpha})^k}{\text{k!}\ \Gamma(\beta k + \gamma)}\right)$$

If we apply the limit for ε → 0 on both sides and since f is continuous, we l et

$$\lim_{\varepsilon \to 0} f\left(\Gamma(\gamma)t\,{}_iS_{p,q}^{a:\beta,\alpha}(\varepsilon t^{-\alpha})\right) = \ f\left(\Gamma(\gamma)t\ \lim_{\varepsilon \to 0} \sum_{n\ =\ 0}^{i} \frac{(a_1\ )_k \ldots (\ a_p)_k}{(c_1\ )_k \ldots (\ c_q)_k} \frac{a^k(\varepsilon t^{-\alpha})^k}{\text{k!}\ \Gamma(\beta k+\gamma)}\right)$$

Because

$$\lim_{\varepsilon \to 0} \sum_{n\ =\ 0}^{i} \frac{(a_1\ )_k \ldots .\ \ (a_p)_k}{(c_1\ )_k \ldots .(\ c_q)_k} \frac{a^k(\varepsilon t^{-\alpha})^k}{\text{k!}\ \Gamma(\beta k + \gamma)} = \frac{1}{\Gamma(\gamma)}$$

we can write.

$$\lim_{\varepsilon \to 0} f\left(\Gamma(\gamma)t\,{}_iS_{p,q}^{a:\beta,\alpha}(\varepsilon t^{-\alpha})\right) = f\ (t)$$

The following theorem is about the basic properties of S-series fractional derivatives.

Theorem 2.

Let α ∈ (0,1], a,b∈ R and f,g S-differentiable functions at a point t > 0. Then(a) iD_S^{α}(af +bg)(t) = aiD_S^{α}f(t) + biD_S^{α}g(t), (b) iD_S^{α}(f .g)(t) = f (t)iD_S^{α}g(t) + f(t)iD_S^{α}g(t), (c) iD_S^{α}($\frac{f}{g}$)(t) = $\frac{g(t)\ \text{i}D_S^{\alpha}\text{f(t)} - \text{f(t)i}D_S^{\alpha}\text{g(t))}}{(g(t))^2}$,(d) If f is differentiable,then i$D_S^{\alpha}(f) =$

$Kt^{1-\alpha}\frac{df(t)}{dt}$ (e) if f ‘ (g(t)) exists, then i$D_S^{\alpha}(fog)(t) =$

$f\quad '(g(t))$i$D_S^{\alpha}\ g(t)$For (e): If g is a constant function in a neighborhood of a. Then clearlyD_S^{α} f(g(a)) = 0. Now, assume that g is not a constant functio n, that is, we can find an ε > 0 for any t_1,t_2∈ (a−ε,a+ε) such that g$(t_1) \neq g(t_2)$ Since g is continuous at a and for small enough ε, wehave

$${}_iD_S^{\alpha}(fog)(a) = \lim_{\varepsilon\to 0}\frac{f\left(g\left(\Gamma(\gamma)\,{}_iS_{p,q}^{\alpha:\beta,\gamma}(\varepsilon t^{-\alpha})\right)\right)-f\ (g(a))}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0}\frac{f\left(g\left(\Gamma(\gamma)\,{}_iS_{p,q}^{\alpha:\beta,\gamma}(\varepsilon t^{-\alpha})\right)\right)-f\ (g(a))}{g\left(\Gamma(\gamma)a\,{}_iS_{p,q}^{\beta,\gamma}(\varepsilon t^{-\alpha})\right)-g(a)}\frac{g\left(\Gamma(\gamma)\,{}_iS_{p,q}^{\alpha:\beta,\gamma}(\varepsilon t^{-\alpha})\right)-g(a)}{\varepsilon}$$

$$= \lim_{\varepsilon_1\to 0}\frac{f\left(g\left(\Gamma(\gamma)\,{}_iS_{p,q}^{\alpha:\beta,\gamma}(\varepsilon t^{-\alpha})\right)\right)-f\ (g(a))}{\varepsilon_1}\lim_{\varepsilon\to 0}\frac{g\left(\Gamma(\gamma)\,{}_iS_{p,q}^{\alpha:\beta,\gamma}(\varepsilon t^{-\alpha})\right)-g(a)}{\varepsilon}$$

= f‘ ((a))i$D_S^{\alpha}g(a)$, with a> 0.Example 3. Now we give the truncated S-series fractional derivatives of some well-known functions by using the result (8). Let n ∈ R and α ∈ (0,1]. Then we h ave the following results

(a)${}_iD_S^{\alpha}(const.) = 0$

(b) ${}_iD_S^{\alpha}(e^{nt}) = \kappa nt^{1-\alpha}e^{nt}$

(c)${}_iD_S^{\alpha}(\sin nt) = knt^{1-\alpha}\cos nt,$ (d)${}_iD_S^{\alpha}(\cos nt) = -\kappa nt^{1-\alpha}\sin nt,$

(e) ${}_iD_S^{\alpha}(t^n) = -\kappa nt^{n-\alpha},$

(f) ${}_iD_S^{\alpha}\left(\frac{t^{\alpha}}{\alpha}\right) = \kappa.$

Theorem 4 (Rolle’s theorem).

Let a > 0 and f : [a, b] → R be a function such that:

(a) f is continuous on [a, b],

(b) f is S-

differentiable on (a, b) for some α ∈ (0,1),

(c) f(a) = f(b).

Then, there exists c ∈ (a,b

), such that ${}_{i}D_{S}^{\alpha}(t^{n})$ f(c) = 0.

Proof. Let f is a continuous function on [a, b] and f(a) = f(b), t

hen there exists a point c ∈ (a, b) at which the function f has a local extreme

. Then, ${}_{i}D_{S}^{\alpha}f\ (c) =$

$$\lim_{\varepsilon\to 0^{-}} \frac{f\left(\Gamma(\gamma)c\ {}_{i}S_{p,q}^{\alpha:\beta,\alpha}(\varepsilon t^{-\alpha})\right)-f\ (c)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0^{+}} \frac{f\left(\Gamma(\gamma)c\ {}_{i}S_{p,q}^{\alpha:\beta,\alpha}(\varepsilon t^{-\alpha})\right)-f\ (c)}{\varepsilon}. = \lim_{\varepsilon\to 0^{\pm}} \frac{{}_{i}S_{p,q}^{\alpha:\beta,\alpha}(\varepsilon t^{-\alpha})}{\varepsilon} = \frac{1}{\Gamma(\gamma)}.$$ th

e two limits have opposite signs. So ${}_{i}D_{S}^{\alpha}$ f(c)=0.

Theorem 5 (Mean value theorem). Let a > 0 and f : [a,b] → R be a functio

n such that:

(a) f is continuous on [a,b];

(

b) f is S-

differentiable on (a, b) for some α ∈ (0,1).

Then, there exists c ∈ (a, b), such that

${}_{i}D_{S}^{\alpha}f\ (c)=$ K$\frac{f(b)-f(a)}{\frac{b^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}}$

$$g\ (t) = f(t) - f(a)\ -\left(\frac{f(b)-f(a)}{\frac{b^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}}\right)\left(\frac{t^{\alpha}}{\alpha}\ -\ \frac{a^{\alpha}}{\alpha}\right)$$

… (5.2.3)

The function g provides the conditions of Rolle's theorem. Then, there exist s a point c ∈ (a,b), such that ${}_iD_S^{\alpha} g\ (c)$ = 0. Applying the new truncated S-series fractional derivative on both sides of the equality (9) and using the pr operties (a) and (f) of Example 1, wehavethe result.

Theorem 6 (Extended mean value theorem). Let f,g: [a,b] → R, a > 0 be t wo functions such that: (a) f, g are continuous on [a,b]; (b) f,g are S-differentiable on (a,b) for some α ∈ (0,1).Then, there exists c ∈ (a,b), such t hat: $\frac{{}_iD_S^{\alpha} f\ (c)}{{}_iD_S^{\alpha} g\ (c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Proof. Consider the following function:

$$F(x) = f(t) - f(a) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)(g(t)-g(a)) \quad \ldots (5.2.4)$$

The function F provides the conditions of Rolle's theorem. Then, there exist s a point c ∈ (a,b), such that ${}_iD_S^{\alpha} f\ (c)$ = 0. Applying the truncated S-series fractional derivative on both sides of the equality (10) and using the p roperty (a) of Example 1, we have the result.

Theorem 7. Let a > 0 and f : [a,b] → R be a function such that: (a) f is cont inuous on [a,b]; (b) f isS-differentiable on (a,b) for some α ∈ (0,1). If for all t ∈ (a,b)${}_iD_S^{\alpha} f\ (t)$ = 0, th en f is a constant function on [a,b].

Proof. Assume that, for all t ∈ (a, b), ${}_iD_S^{\alpha} f\ (t)$ = 0, and let, t_1, t_2∈ [a,b], wi th$t_1 < t_2$. Since f is also continuous in $[t_1, t_2]$ and S-differentiable in (t_1, t_2), from Rolle's theorem, there exists a point c ∈ (t_1, t_2) with

${}_iD_S^{\alpha} f\ (c)$= K$\frac{f(t_2)-f(t_1)}{\frac{t^{\alpha}{}_2}{\alpha} - \frac{{t_1}^{\alpha}}{\alpha}} = 0$.So, f($t_1$) = f($t_2$). Since $t_1 <$ t_2 are arbitrary chosen from [a,b], f has to be a constantfunction.

Corollary 8. Let a > 0 and f, g : [a, b] → R be functions such that for all α ∈ (0,1) and t ∈ (a,b), ${}_iD_S^{\alpha}$f(t) = ${}_iD_S^{\alpha}$g(t).

Then, there exists a constant c such that f(t) = g(t) +c Proof. Apply Th eorem 7 with choosing h(t) = f(t)− g(t).

Theorem 9. Let K > 0 and f : [a,b] → R be a function which continuous on [a,b] and S-

differentiable on (a,b) for some α ∈ (0,1). Then, for all t ∈ (a,b) • if ${}_iD_S^{\alpha}$ f(t) > 0, then f is increasing on [a,b], • if ${}_iD_S^{\alpha}$f(t) < 0, then f is decreasing on [a ,b]. Proof. From Theorem 7 we know that for t_1,t_2∈ [a,b] there exist a c ∈ (t_1, t_2) such as${}_iD_S^{\alpha} f\ (c)$ = K$\frac{f(t_2)-f(t_1)}{\frac{t^{\alpha}{}_2}{\alpha}-\frac{t_1{}^{\alpha}}{\alpha}}$If ${}_iD_S^{\alpha}$ f(c) > 0 then f(t_2) > f(t_1)while $t_2 >$ t_1, so f is increasing since t_1 and t_2 chosen arbitrarily. But if ${}_iD_S^{\alpha}$ f(c) < 0 th en f(t_2) > f(t_1) while t_2<t_1(or (t_2) > f(t_1)while $t_2 > t_1$ so f is decreasing.

Theorem 10. Let K > 0 and f,g : [a,b] → R be functions which continuous on [a,b], S-

differentiable on (a,b) for some α ∈ (0,1) and for all t ∈ [a,b], ${}_iD_S^{\alpha}$ f(t) ≤ ${}_iD_S^{\alpha}$ g(t). Then, • if f(a) = g(a), then f(t) ≤ g(t) for all t ∈ [a, b],

• if f(b) = g(b),

then f(t) ≥ g(t) for all t ∈ [a, b]. Proof. The proof is trivial when you conside r the function h(t) = g(t)− f(t).

Theorem 11. Let f : [0,∞) → R be a two times differentiable function with t > 0 and$\alpha_1, \alpha_2 \alpha_1, \alpha_2$∈ (0,1). Then ${}_iD_S^{\alpha_1+\alpha_2}$ f(t) ≠

${}_iD_S^{\alpha_1}$(${}_iD_S^{\alpha_2}$f) (t).Proof. From the equality (8) we have ${}_iD_S^{\alpha_1+\alpha_2}$ f(t) = K $t^{1-\alpha_1-\alpha_2}$ f ' (t)… (5.2.5) but for the other side we have${}_iD_S^{\alpha_1}$(${}_iD_S^{\alpha_2}$f) (t) = ${}_iD_S^{\alpha_1}$(K$t^{1-\alpha_2}$ f ' (t))

$= K^2 t^{1-\alpha_1} t^{1-\alpha_2} f(\quad '(t))' = K^2 t^{1-\alpha_1-\alpha_2}(1-\alpha_2) f(\quad '(t) + t f''(t))$

… (5.2.6)

The proof is clear from (11) and (12). The following result is the direct cons equences of the previoustheorem.

Corollary 12. Let f : [0,∞) → R be a two times differentiable function with t > 0 and $\alpha_1, \alpha_2 \in$ (0,1). Then ${}_iD_S^{\alpha_1}({}_iD_S^{\alpha_2}$f) (t) ≠ ${}_iD_S^{\alpha_2}($ ${}_iD_S^{\alpha_1}$ f) (t). The following definition is about the S-series fractional derivative operator for α ∈ (n,n+1],n∈ N.∈N.

Definition 3. Let α ∈ (n,n + 1], n ∈ N and for t > 0, f be an times differentiable function. The truncated S-series fractional derivative of order α off is given as

$${}_iD_M^{\alpha:n} \text{ f(t)} := \lim_{\varepsilon \to 0} \frac{f\left(\Gamma(\gamma)t \ {}_iS_{p,q}^{\alpha:\beta,\alpha}(\varepsilon t^{n-\alpha})\right) - f^{(n)}(c)}{\varepsilon}$$

... (5.2.7)

(13) if and only if the limit exists.

Remark 1. For t > 0, α ∈ (n,n+1] and for (n+1) times differentiable function f , it is easy to show that ${}_iD_S^{\alpha:n}$ f(t) = K$t^{n+1-\alpha} f^{(n+1)}$(t) by using (13), (8) and induction on n.**S-series** Fractional Integral In this section, we defined the corresponding S-series fractional integral operator I α S f(t). We want that our integral operat

or to satisfies $_iD_S^\alpha(J_S^\alpha$ f (t)) = f (t).

Let F(t) = J_S^α f (t) be a differentiable function, then from (8), we have the fo

llowing differential equation f (t) =$_iD_S^\alpha$ (F(t)) = K$t^{1-\alpha}\frac{dF(t)}{dt}$ which have a s

olution of the form for

$$a_n \neq 0,\ (\text{n} = 1,2,\ \ldots.\ \text{P})\text{F (t)}= K^{-1}\int\frac{f(t)}{t^{1-\alpha}}dt$$

….5.2.8

This yields the following definition.

Definition 4. Let a ≥ 0 and t ≥ a, and f is defined in (a,t]. If the following i

mproper Riemann integral exists, then for α ∈ (0,1), the α order S-

series fractional integral of a function f is defined by$J_S^\alpha f(t)$: =

$J_S^\alpha\begin{bmatrix} a_1, \ldots, & a_p \\ c_1, \ldots, & c_q \end{bmatrix}\beta,\gamma\Big]$f (t) = $K^{-1}\int\frac{f(t)}{t^{1-\alpha}}dt$,...(5.2.9)where the conditions are the

same as (7) with $a_n \neq$

0, n = 1,2,., p.Remark 2.It can easily be seen from the definition of S-

series fractional integral that, the integral operator is linear and $J_S^\alpha f(a)$ = 0.

For the rest of the paper, we assume that $a_n \neq 0$,

n = 1,2 ,..., p.

Theorem 13. Let a ≥ 0, α ∈ (0,1) and f is a continuous function such that J_S^α

f(t) exists. Then for t ≥ a, $_iD_S^\alpha$ (J_S^αf(t)) = f(t).

Proof. Since f is continuous, J_S^{α} f(t) is differentiable. Then from (8), we have

${}_iD_S^{\alpha}$ (J_S^{α}f(t)) = K$t^{1-\alpha}\frac{d}{dt}J_S^{\alpha}$f(t)

=

$$t^{1-\alpha}\frac{d}{dt}\left(\int_a^t \frac{f(t)}{t^{1-\alpha}}dt\right) \quad = \text{ f (t).}$$

which completes the proof

Theorem 14. Let f : (a, b) → R be a differentiable function and α ∈ (0,1]. T hen, for all t > a, we have J_S^{α}(${}_iD_S^{\alpha}$f(t)) = f(t)− f(a). Proof. Since the functio n f is differentiable, by using the fundamental theorem of calculus for the in teger-

order derivatives and (8), we get J_S^{α}(${}_iD_S^{\alpha}$f(t)) = $K^{-1}\int_a^t \frac{{}_iD_S^{\alpha}f(t)}{t^{1-\alpha}}d\,x$=

$\int_a^t \frac{df(t)}{dt}d\,x$=f(t) – f (a) . which gives the result.

Remark 3. If f(a) = 0 then J_S^{α}(${}_iD_S^{\alpha}$ f(t)) =${}_iD_S^{\alpha}$(J_S^{α}f(t)) = f(t).

Theorem 15. Let f : [a, b] → R be a continuous function with 0 < a < b and α ∈ (0,1). Then for K > 0 we have $\left|J_S^{\alpha}\ f(t)\right| \leq$ $J_S^{\alpha}|f|(t)$.Proof. From the definition of S – series fractional integral we have

$\left|J_S^{\alpha}\, f(t)\right|$ = $\left|K^{-1}\int_a^t \frac{f(x)}{x^{1-\alpha}}d\,x\right| \leq |K^{-1}| \left|\int_a^t \frac{f(x)}{x^{1-\alpha}}d\,x\right| \leq$

$K^{-1}\left|\int_a^t \frac{f(x)}{x^{1-\alpha}}d\,x\right|$

$= K^{-1}\int_a^t \frac{f(x)}{x^{1-\alpha}}d\,x,$ wh

ich completes the proof.

Corollary 16. Let f : [a, b] → R be a continuous function such that N = $\sup_{t\in[a,b]}$ | f(t)|.

Then, for all t ∈ [a, b] with 0 < a < b, α ∈ (0,1) and K > 0 we have |J_S^{α} f(t)| ≤ K^{-1}N $\left(\frac{t^{\alpha}}{\alpha} - \frac{t^{\alpha}}{\alpha}\right)$. Proof. From the previous theorem we have

$\left|J_S^{\alpha}\, f(t)\right| \leq$

$J_S^{\alpha}|f|(t).$ = $K^{-1}\left|\int_a^t \frac{|f\ (x)|}{x^{1-\alpha}}d\,x\right|$

$= K^{-1}N\ \int_a^t x^{\alpha-1}\,dx.$

Which gives the result.

Theorem 17. Let f,g:[a,b] → R be two differentiable functions and α ∈ (0,1). Then ˆ $\int_a^b f\ (t)\,{}_iD_S^{\alpha}\,g(t)\ d_{\alpha}t$=

f(t)g(t)$I_a^b - \int_a^b g(t)\ {}_iD_S^{\alpha}\ f(t)d_{\alpha}t,$ where $d_{\alpha}t = K^{-1}t^{\alpha-1}dt$..

Proof. Using the definition of S-

series fractional integral (14), (8) and applying the fundamental theorem of calculus for integer-order derivatives, we get

$\int_a^b f\ (t)\,{}_iD_S^{\alpha}\, g(t)\ d_\alpha t\ = K^{-1}\int_a^t \frac{f\ (t)}{t^{1-\alpha}}\,{}_iD_S^{\alpha}\ \ g(t)d\,t = \int_a^b f\ (t)\frac{dg(t)}{dt}\ dt\ =$ f(t

)g(t) $I_a^b -$

$\int_a^b g(t)\frac{d\ f(t)}{dt}dt$

$= \text{f(t)g(t)}\ I_a^b - \int_a^b g(t)\ {}_iD_S^{\alpha}\ f(t)d_\alpha t.$

Which completes the proof. Now we define the S-

series fractional integral for α ∈ (n, n+1] as follows.

Definition 5. Let a ≥ 0 and t ≥ a, and f is defined in (a,t]. If the following i

mproper Riemann integral exists, then for α ∈ (n,n+1), the α order S-

series fractional integral of a function f is defined by

$$J_S^{\alpha:n} f(t):\ = J_S^{\alpha:n}\begin{bmatrix} a_1, \dots,\ a_p \\ c_1, \dots,\ c_q \end{bmatrix}\beta,\gamma \Big]\text{f (t)}\ \ = K^{-1}\int_a^t dt\int_a^t dt \dots\ \int_a^t \frac{f(t)}{t^{n+1-\alpha}}dt,. \qquad \dots(5.2.9)$$

where the conditions are the same as (7) with $a_n \neq 0$, n = 1,2,..., p.

The following theorem is a generalization of

Theorem 18. Let α ∈ (n, n+1] and f : [a,∞)⟶ R be (n+1) times differentiabl

e function for t > a. Then we have $J_S^{\alpha:n}(\ {}_iD_S^{\alpha:n}$ f) (t) = f (t) -

$\sum_{k=0}^{n}\frac{f^{(k)}\ (a)(t-a)^k}{k\ !}$

Proof. From (7) and (15) we have

$$J_S^{\alpha:n}({}_iD_S^{\alpha:n} \text{ f}) (t) = K^{-1} \int_a^t dt \int_a^t dt \ldots \int_a^t \frac{{}_iD_S^{\alpha:n} f(t)}{t^{n+1-\alpha}} dt=$$

$$\int_a^t dt \int_a^t dt \ldots \int_a^t f^{(n+1)} (t)dt.$$

Which gives the result.

3. Applications to S-series Fractional Differential Equations

In this section, we obtained the general solutions of linear fractional d ifferential equations including **the S-**

series fractional derivative operator.Example 19. Let u = u(t) is a M-differentiable function and assume that for α ∈ (0,1] the linear M-series fractional differential equation

$${}_iD_M^{\alpha} u(t) + p(t)u(t) = q(t)$$

…. (5.3.1)

is given. If u is also a differentiable function then by using (8), we get a line ar ordinary differential equation

$$\frac{du(t)}{dt} + K^{-1}t^{\alpha-1}p(t)u(t) = K^{-1}t^{\alpha-1}q(t).$$

The integrating factor of the equation can be found as μ(t) = $e^{K \int t^{\alpha-1}} p(t)dt$, which yields the solution

$$u(t) = e^{-K-1 \int \frac{p(t)}{t^{1-\alpha}} dt} \left[K^{-1} \int \frac{q(t)}{t^{1-\alpha}} e^{K-1 \int \frac{p(t)}{t^{1-\alpha}} dt} \; dt + C \right],$$

where C is a constant. By definition of the M-

series integral operator we can write the last equality as

$$u(t) = e^{-J_M^{\alpha}\, p(t)}[\, J_M^{\alpha}\big(q(t)e^{-J_M^{\alpha}\, p(t)}\big) + C],$$

$$\text{….}(5.3.2)$$

If we choose p(t) = −λ, q(t) =0, then the linear M-

series fractional differential equation(16) turns${}_{i}D_M^{\alpha}$u(t)= $\lambda u(t)$.and the gener

al solution can be found from (17) as u(t) = C $e^{-K-1\frac{\lambda}{\alpha}t^{\alpha}}$. Since $e^t =$

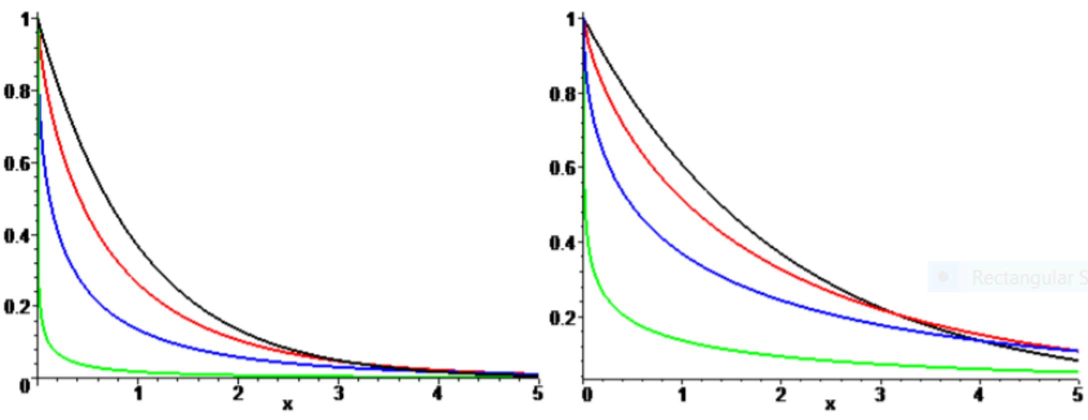
${}_{\infty}M_{1.1}^{1.1}(t)$. we can write the solution using truncated M-

series as u(t) = C${}_{\infty}M_{1.1}^{1.1}\left(-K^{-1}\frac{\lambda}{\alpha}t^{\alpha}\right)$.

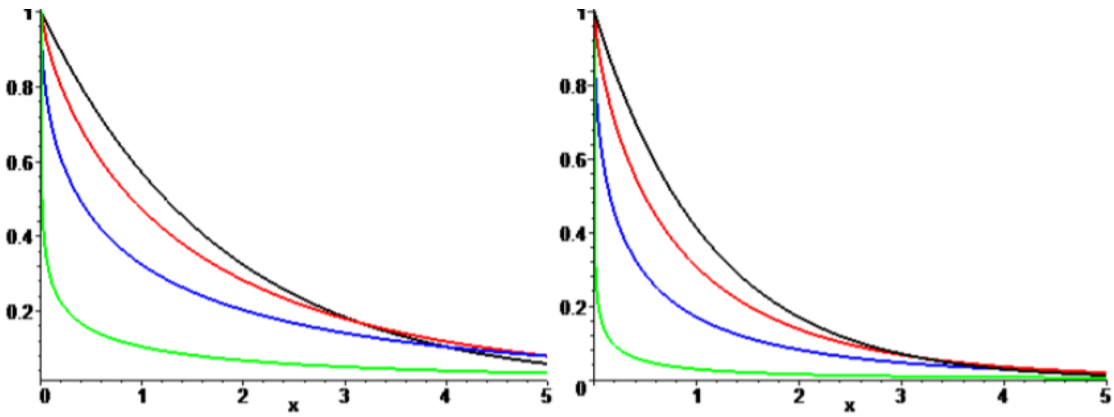
For the fixed values a_n = 1, c_m = 1, (n = 1,2,..., p; m = 1,2,...,q), this result

coincides with the results given in [27] when λ = 1 and coincides with the c

orresponding integer-order result when α = β = λ = 1,a=1 and k!=1



$\beta = 1.0$ *and* $\gamma = 1.0\beta = 1.0$and $\gamma = 0.5$



$\beta = 0.5 \; and \; \gamma = 0.5 \beta = 0.5 \; and \; \gamma = 1.0$

Fig. 1 The graphs of (18) from α = 0.25 (green) to α = 1.00 (black) by step size 0.25.

In the following, the reader can find the graphs of the solution function (18) for different α,β and γ values with the fixed values C = λ = 1 and a_n = 1, c_m = 1, (n = 1,2,..., p; m = 1,2,...,q).

4.Concluding Remarks and Observations:

In this paper, we first presented a fractional derivative operator, which is also a generalization of truncated M-fractional derivative, by using generalized S-series. Then we definedthe corresponding integral operator. Unlike fractional operators with different kernels, we showed that there are many common properties provided by both these and the corresponding integer-order operators. We also used these operators in differential equation problems as applications These problems are hard to solve using the classical definitions of fractional derivatives. Besides, from equality (e) of Example 1, we observed that, for polynomials, truncated M-series fractional derivative coincides with the Riemann-

Liouville and Caputo fractional derivatives [20] up to a constant multiple. In this case, we can say that the truncated S-series fractional derivative operator can be used instead of Riemann-Liouville or Caputo type derivatives (and also their generalizations) to solve some difficult problems. Our definition is also a generalization of the M-fractional derivative for p = q = 1 which defined in [38]. It is also possible to define new fractional derivatives by using other special functions instead of S-series. Since S-series is a general class of special functions, all future definitions have a chance to be the special cases of our definition.

Chapter-6

APPLICATIONS OF FRACTIONAL CALCULUS IN MECHANICAL ENGINEERING

6.1 Introduction:

This chapter provides a study offractional mechanics, where the time derivative is replaced with a fractional derivative of order α. We then solve some simple fractional differential equations of mechanics.

In fractional mechanics, Newton's second law of motion becomes F = ma = m$D_*^{\alpha}v,$ where m is the mass of the body in motion. When the force is const ant, the body moves with a constant fractional acceleration of $\frac{F}{m}$. Now consi der the vertical motion of a body in a resisting medium in which there exists aresisting force proportional to the fractional velocity, as is sometimes the c ase with viscoelastic drag in certain types of materials like polymers [15]. We assume the body is projected downward with zero initial velocity (v(0) = 0) in a uniform gravitational field. For some constant k denoting the rat io of resistance to fractional velocity

, the equationofmotion is given by :

$$F = mD_*^{\alpha+2}v = mg - kv \ldots(6.1.1)$$

Applying a fractional integral of degree αin (6.1.1) and dividing both sides by m we get :

$$v\,(t) = gJ^{\alpha+2}[1] - \frac{k}{m}J^{\alpha+2}[v(t)] \ldots(6.1.2)$$

Multiply both sides by $(-\frac{k}{m})^n J^{n(\alpha+2)}$ $in\ equation$ (6.1.2)and sum both s ides for n from 0 to ∞we get

$$\sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{n} J^{n(\alpha+2)}\ v(t) = \mathrm{g}\sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{n} J^{n(\alpha+2)} J^{\alpha+2}[1] -$$

$$\sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{n} J^{n(\alpha+2)} \frac{k}{m} J^{\alpha+2} v(t)$$

$$\sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{n} J^{n(\alpha+2)}\ v(t) = \mathrm{g}\sum_{n=0}^{\infty}(-\frac{k}{m})^{n} J^{(n+1)(\alpha+2)}\,[1] +$$

$$\sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{(n+1)} J^{(n+1)(\alpha+2)}\ v(t) \sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{n} J^{n(\alpha+2)}\ v(t) -$$

$$\sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{(n+1)} J^{(n+1)(\alpha+2)}\ v(t) = \mathrm{g}\sum_{n=0}^{\infty}(-\frac{k}{m})^{n} J^{(n+1)(\alpha+2)}\,[1]$$

The two sums on the left cancel each other out except for the n = 0 cases,

giving v (t) = $\mathrm{g}\sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{n} J^{(n+1)(\alpha+2)}$ (1)

We know that$J^{n\alpha}[1] =$

$\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$. Replacing n with n+1 and plugging this into our formula we get:

$$\mathrm{v}\ (\mathrm{t}) = \mathrm{g}\sum_{n=0}^{\infty}\left(-\frac{k}{m}\right)^{n} \frac{t^{(n+1)(\alpha+2)}}{\Gamma((n+1)(\alpha+2)+1)} =$$

$$\frac{mg}{k}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{k}{m}\right)^{n} \frac{t^{(n+1)(\alpha+2)}}{\Gamma((n+1)(\alpha+2)+1)} \ldots(6.1.3)$$

which is the required result $if\ we\ put\quad n = n+1,\ \ \alpha = \alpha + 2$then

$$\mathrm{V(t)} = \frac{mg}{k}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{k}{m}\right)^{n}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \frac{mg}{k}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{k}{m}t^{\alpha}\right)^{n}}{\Gamma(n\alpha+1)}$$

$$= \frac{mg}{k}\left[1 - \sum_{n=0}^{\infty}\frac{\left(-\frac{k}{m}t^{\alpha}\right)^{n}}{\Gamma(n\alpha+1)}\right] = \frac{mg}{k}\left[1 - E_{\alpha,1}\left(-\frac{k}{m}t^{\alpha}\right)\right] \ldots(6.1.4)$$

And so we have found a solution for v(t). This problem was solved in a paper by Jung and Chung [13].

6.2 Now consider the fractional harmonic oscillator problem: Thefractional harmonic oscillator problem is given as under:

$$\mathrm{m}D_*^{\frac{1}{2}\alpha}\mathrm{x(t)} = -\ \mathrm{m}\omega^2 x(t) \text{with the initial condition: } \ldots(6.2.1)$$

x(0) = A, $(D_*^{\alpha/2}x)$ (0) = v_0

Applying a fractional integral of order $3\alpha/2$ to the equation (6.2.1) and dividing both sides by m, we get: D_*^{α}x(t) - v_0 = - $\omega^2 J^{3\alpha/2}x(t)$

Move v_0 *to the other side and integrate once and we get*:

$$\mathrm{x(t)} - \mathrm{A} = v_0 J^{\alpha}1 - \omega^2 J^{\frac{\alpha}{2}}x(t) \ldots(6.2.2)$$

Now multiply both sides of the equation (6.2.2) by $(-\ \omega^2)^m J^{2m\alpha}$ to get:$(-\ \omega^2)^m J^{2m\alpha}$x(t) $-(-\omega^2)(-\ \omega^2)^m J^{2(m+1)\alpha}$

$x(t)$ = $v_0(-\ \omega^2)^m J^{\frac{(4m+1)\alpha}{2}}[1] + A(-\ \omega^2)^m J^{2m\alpha}[1]$

Now summing both sides form from 0 to ∞ *yields*:

$$\sum_{m=0}^{\infty}(-\ \omega^2)^m J^{2m\alpha}\mathrm{x(t)} - \sum_{m=0}^{\infty}(-\ \omega^2)^{m+1} J^{2(m+1)\alpha}x(t) = v_0 \sum_{m=0}^{\infty}(-\ \omega^2)^m J^{(4m+1)\alpha/2}[1] + A\sum_{m=0}^{\infty}(-\ \omega^2)^m J^{2m\alpha}[1]$$

x(t) +$\sum_{m=1}^{\infty}(-\omega^2)^m J^{2m\alpha}x(t)$-

$\sum_{m=1}^{\infty}(-\omega^2)^{m+1}J^{2m\alpha}x(t)$ = $v_0\sum_{m=0}^{\infty}(-\omega^2)^m J^{(4m+1)\alpha/2}$ [1] +

$A\sum_{m=0}^{\infty}(-\omega^2)^m J^{2m\alpha}[1]$ x(t) =

$v_0\sum_{m=0}^{\infty}(-\omega^2)^m J^{(4m+1)\alpha/2}[1] + A\sum_{m=0}^{\infty}(-\omega^2)^m J^{2m\alpha}[1]$

$= v_0\sum_{m=0}^{\infty}(-\omega^2)^m \frac{t^{(4m+1)\alpha/2}}{\Gamma((4m+1)\alpha/2+1)} +$

$A\sum_{m=0}^{\infty}(-\omega^2)^m \frac{t^{2m\alpha}}{\Gamma(2m\alpha+1)}$ =

$A\sum_{m=0}^{\infty}\frac{(-1)^m(\omega t^\alpha)^{2m}}{\Gamma(2m\alpha+1)} + \frac{v_0}{\omega^{1/2}}\sum_{m=0}^{\infty}\frac{(-1)^m(\omega t^\alpha)^{(4m+1)/2}}{\Gamma((4m+1)\alpha/2+1)}$

= A$C_{\alpha,1}(\omega t^\alpha)$+ $\frac{v_0}{\omega^{1/2}}S_{\alpha,1}(\omega t^\alpha)$ where $C_{\alpha,1}$ and $S_{\alpha,1}$ are the Mittag-Leffler cosine and sine functions:$C_{\alpha,1}(x) = \frac{(-1)^m(x)^{2m}}{\Gamma(2m\alpha+1)}$, $S_{\alpha,1}(x) = \frac{(-1)^m(x)^{(4m+1)/2}}{\Gamma((4m+1)\alpha/2+1)}$Mittag-Leffler cosine and sine functions can also be written as: $C_{\alpha,1}(x) = \frac{1}{2}\left[E_{\alpha,1}(ix) + E_{\alpha,1}(-ix)\right]$,$S_{\alpha,1}(x) = \frac{1}{2}\left[E_{\alpha,1}(ix) - E_{\alpha,1}(-ix)\right]$ These formulas are analogous to the formulas for cosine and sine in terms of e^{ix} and e^{-ix}. [13]This completes the analysis.

Conclusion: The applications of fractional calculus can be seen in many are as of engineering and sciences.It has been played an important role in Mech anical engineering. In this chapter, we have obtained the closed-form solution of fractional differential equation associated with Newton'sla w of fractional order and fractional harmonic oscillator problemin terms of Mittag-Leffler function The results are obtained in [13] are specialcases of ourresul t.

Chapter 7

New Approach of Derivative of Arbitrary order without Singular Kernel

7.1Introduction:

In this endeavor, a new approach of the derivative of arbitrary order (FD) with the kernel of the smooth type that gains different depictions for the temporal and spatial variables has been given. It first applies to the time variables and hence it is fit to use transform of Laplace type (LT). Secondly, a definition is linked to the spatial type variables, by a global derivative of arbitrary order (FD), for which we will apply the transform of Fourier type (FT). The courtesy for this new methodology with a kernel of regular type was native from the vision that there is a period of global systems, which can designate the material heterogeneities and the fluctuations of unlike scales, which cannot be well described by traditional local theories or by arbitrary order models with the kernel of singular type.

7.2 A new fractional time derivative:

Let us recall the usual Caputo fractional time derivative (UFD_t) of order α, given by

$$D_t^{(\alpha)} f\ (t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'\ (\tau)}{(t-\tau)} d\tau \qquad \ldots (7.2.1)$$

with $\alpha \in [0,1]$ and $a \in [-\infty, t)$, $f \in H^1$ (a,b), b > a. By changing the kernel $(t-\tau)^{-\alpha}$ with the function exp $(-\frac{\alpha}{1-\alpha}$ t) and $\frac{1}{\Gamma(1-\alpha)}$with $\frac{M(\alpha)}{1-a}$, and we replace exponential function by Mittag-Leffler function we obtain the following new definition of fractional time derivative NFD

$$D_t{}^{(\alpha)}f\ (t) = \frac{M(\alpha)}{(1-\alpha)}\int_{-\infty}^{t} f'(\tau)\, E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \qquad \ldots\ (7.2.2)$$

where M(α) is a normalization function such that M(0) = M,(1) = 1. According to the definition (7.2.2), the NFD_t is zero when f(t) is constant, as in the NFD_t , but, contrary to the UFD_t, the kernel does not have singularity for t = τ. The new NFDt can also be applied to functions that do not belong to H^1 (a, b). Indeed, the definition (7.2.2) can be formulated also for f $\in L^1$ (−∞, b) and for any α ∈ [0,1] as

$$D_t{}^{(\alpha)}f\ (t) = \frac{\alpha M(\alpha)}{(1-\alpha)}\int_{-\infty}^{t} (f(t) - f(\tau))E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau$$

Now, it is worth observing that if we put

σ = $\frac{1-\alpha}{\alpha}$∈ [0, ∞] , α = $\frac{1}{1+\sigma}$∈ [0,1] the definition (7.2.2) of NFD_t assumes the form

$$D_t{}^{(\alpha)}f\ (t) = \frac{N(\sigma)}{\sigma}\int_{a}^{t} f'(\tau)\, E_\beta\left[-\frac{(t-\tau)}{\sigma}\right] d\tau \qquad \ldots\ (7.2.3)$$

where σ ∈ [0,∞] and N(σ) is the corresponding normalization term of M(α), such that N(0) = N(∞) = 1. Moreover, because

$$\lim_{\sigma\to 0,\quad \beta\to 1} \frac{1}{\sigma} E_\beta\left[-\frac{(t-\tau)}{\sigma}\right] = \delta(t-\tau) \quad \ldots\ (7.2.4)$$

and for $\alpha \to 1,\ we\ have\ \sigma \to 0$.

Then (see [35] and [36])

$$\lim_{\substack{\alpha\to 1\\ k\to 1}} D_t{}^{(\alpha)}f\ (t) = \frac{M(\alpha)}{(1-\alpha)}\int_{a}^{t} f'(\tau))E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \qquad \ldots\ (7.2.5)$$

$$\lim_{\substack{\alpha\to 1\\ k\to 1}} D_t{}^{(\alpha)}f\ (t) = \lim_{\sigma\to 0}\frac{N(\sigma)}{\sigma}\int_{a}^{t} f'(\tau))E_\beta\left[-\frac{(t-\tau)}{\sigma}\right] d\tau$$

Otherwise, when α → 0, k→ 1 then σ → +∞. Hence

$$\lim_{\alpha\to 0} D_t^{(\alpha)} f\ (t) = \lim_{\alpha\to 0} \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(\tau)) E_\beta \left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau$$

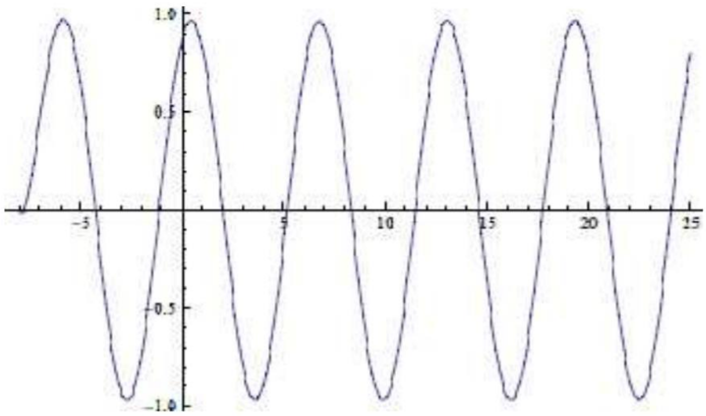
$$\lim_{\alpha\to 1} D_t^{(\alpha)} f\ (t) = \lim_{\sigma\to \infty} \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) E_\beta \left[-\frac{(t-\tau)}{\sigma}\right] d\tau$$

= f (t) – f (a) … (7.2.6)

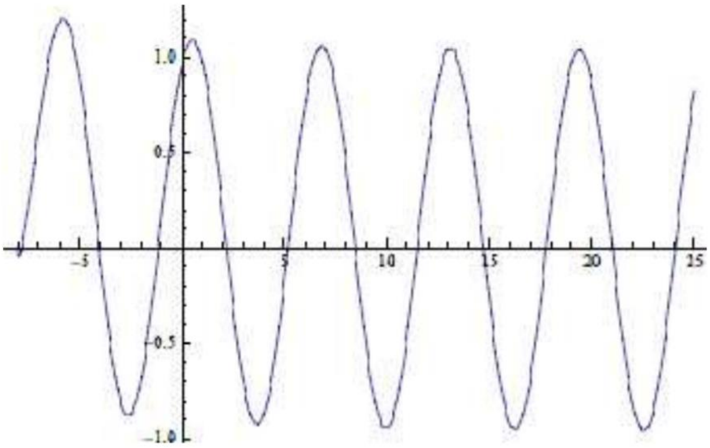
Let us consider, the NFD_t of a particular function, as f(t) = sinωt, for α = 0. 65, a = −8 and ω = 1, when be take β =1 the graph is similar to Caputo an d Fabrizio,

$D_t^{0.65} \sin\omega t = \frac{M\ (0.65)}{0.35} \int_a^t cos\tau\ E_\beta[-2(t-\tau)]\, d\tau \dots$ (7.2.7)

The simulation of this derivative produces the following pictures

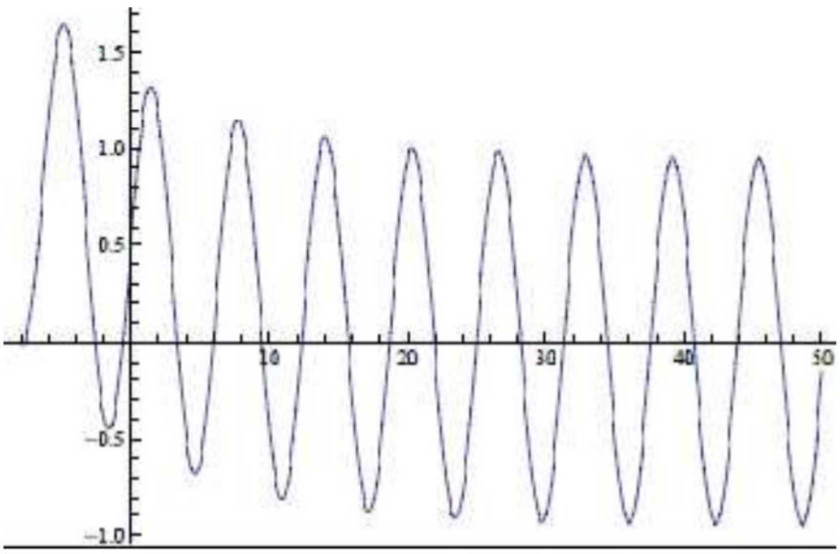


(1). Simulation of NFD_t (6.1.7), with α = 0.65 in the time interval [−8,25]

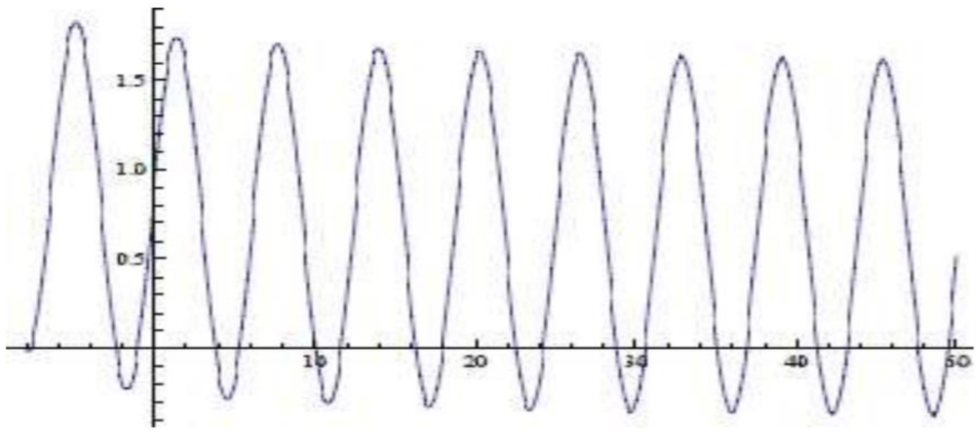


(2). Simulation of UFD_t (2.1), with α = 0.65 in the time interval [−8,25]

From these two simulations with α = 0.65, it appears as the classical NFD_t is very similar to the UFD_t . Otherwise, when we study models with α clos e to 0, we see a different behavior



(3). Simulation of NFD_t (2.7) with α = 0.1 in the time interval [−8,50]



(4). Simulation of UFD_t (2.1), with α = 0.1 in the time interval [-8, 50]

So that, for α = 0.1 in Fig.3 and Fig. 4 we observe different actions between $NFD_t$and UFD_T simulations. In particular, the classical UFDt is more affe cted by the past, compared with the NFD which shows a rapid stabilization. If n ≥ 1, and α ∈ [0,1] the fractional time derivative$D_t{}^{\alpha+n}$ f(t) of order (n+α) is definedby

$$D_t{}^{\alpha+n} f\ (t) = D_t{}^{\alpha} f\ (t)\left(D_t{}^{n} f\ (t)\right)\ldots \quad (7.2.8)$$

Theorem 1. For NFD_t , if the function f(t) is such that $f^{(s)}$ (a) = 0 ,

s = 1,2,..,n

then, we have

$$D_t{}^{n}\left(D_t{}^{\alpha} f\ (t)\right) = D_t{}^{\alpha}\left(D_t{}^{n} f\ (t)\right) \quad \ldots (7.2.9)$$

Proof. We begin considering n = 1, then from definition (2.8) of $D_t{}^{(\alpha+1)}$ f(t), we obtain

$$D_t^{\alpha}(D_t^{1}f\ (t)) = \frac{M(\alpha)}{(1-\alpha)}\int_a^t f(\tau)E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau \qquad \ldots\ (7.2.10)$$

Hence, after integration by parts and assuming f ′ (a) = 0, we have

$$D_t^{\alpha}(D_t^{1}f\ (t)) = \frac{M(\alpha)}{(1-\alpha)}\int_a^t (\frac{d}{d\tau}f(\tau))E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau$$

$$= \frac{M(\alpha)}{(1-\alpha)}\left[\int_a^t \frac{d}{d\tau}f(\tau)E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau - \frac{\alpha}{1-\alpha}\int_a^t f(\tau)E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right]$$

$$= \frac{M(\alpha)}{(1-\alpha)}\left[\ f(t) - \frac{\alpha}{1-\alpha}\int_a^t f(\tau)E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right]$$

Otherwise

$$D_t^{1}(D_t^{\alpha}f\ (t)) = \frac{d}{dt}\ (\frac{M(\alpha)}{(1-\alpha)}\int_a^t f(\tau)E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau)$$

$$= \quad \frac{M(\alpha)}{(1-\alpha)}\left[\ f(t) - \frac{\alpha}{1-\alpha}\int_a^t f(\tau)E_\beta\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right]$$

It is easy to generalize the proof for any n > 1.

In the following, we suppose the function M(α) = 1.

7.3 The Laplace transform of the NFD_t

To study the properties of the NFD_t, defined in equation (2.3) with a = 0, h as priority the computation of its Laplace transform (LT) given with p varia ble

$$\text{LT}\left[D^{(\alpha)}{}_t\ f(t)\right] = \frac{1}{(1-\alpha)}\int_0^\infty E_\beta - pt\int_0^t f(\tau)\ E_\beta - \frac{\alpha(t-\tau)}{(1-\alpha)}\ d\tau dt$$

Hence, from the property of Laplace transform of convolution, we have

$$\text{LT}\left[D^{(\alpha)}{}_t\ f(t)\right] = \frac{1}{(1-\alpha)}LT\ \left(f(t)\right)LT\left(E_\beta - \frac{\alpha t}{(1-\alpha)}\right)$$

$$= \qquad \frac{(pLT\ \left(f(t) - f(0)\right)}{p+\alpha(1-p)}$$

Similarly

$$\text{LT}\left[D^{(\alpha+1)}{}_t\ f(t)\right] = \frac{1}{(1-\alpha)} LT\ \ \overline{(f}\ (t))\ LT(E_\beta - \frac{\alpha t}{(1-\alpha)}\ )$$

$$= \frac{\left(p^2 LT\ [f(t)] - pf(0) - f'(0)\right)}{p + \alpha(1-p)}$$

Finally,

$$\text{LT}\left[D^{(\alpha+n)}{}_t\ f(t)\right] = \frac{1}{1-\alpha}\ LT\left[f^{(n+1)}(t)\right] LT\left[E_\beta - \frac{\alpha t}{(1-\alpha)}\right]$$

$$= \frac{p^{n+1} LT\ [f(t)] - p^n f(0) - p^{n-1} f'(0) \ldots\ f^{(n)}(0)}{p + \alpha(1-p)}$$

7.4 Fractional gradient operator

In this section, we introduce a new notion of fractional gradient able to desc

ribe non-

local dependence in constitutive equations (see [37] and [38]). Let us consid

er a set Ω ∈R^3 and a scalar function

u(·) : Ω → R, we define the fractional gradient of order α ∈ [0,1] as follows

$$\nabla^{(\alpha)}\ u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega^{\cdot} \nabla u(y)\ E_\beta \left[-\frac{\alpha^2 (x-y)^2}{(1-\alpha)^2}\right] dy \ldots \quad (7.4.1$$

with x,y∈ Ω

It is simple to prove from definition (4.1) and by the property

$\lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} E_{\beta}\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] = \delta(x-y)$ that

$\nabla^{(1)}$ u(x) = ∇ u(x) and $\nabla^{(0)}$ u(x) = $\int_{\Omega}^{\cdot} \nabla u(y) dy$

So, when α = 1, $\nabla^{(1)}$u(x) loses the non-locality, otherwise $\nabla^{(0)}$u(x) is related to the mean value of ∇u(y) on **Ω**. In the case of a vector u(x), we define the fractional tensor by

$$\nabla^{(\alpha)}u(x)= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega}^{\cdot} \nabla . u(y)\, E_{\beta}\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy \ldots \quad (7.4.2)$$

Thus, a material with the non-local property may be described by fractional constitutive equations. As an example, we consider an elastic non-local material, defined by the following constitutive equation between the stress tensor T and $\nabla^{(\alpha)}$u(x)T (x, t) = A$\nabla^{(\alpha)}$u (x, t) , α ∈ (0,1]

where A is a fourth order symmetric tensor, or in the integral form

$$T(x,t) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega}^{\cdot} \nabla u(y)\, E_{\beta}\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy$$

Likewise, we introduce the fractional divergence, defined for a smooth

u(x): Ω →R^3 by

$$\nabla^{(\alpha)}u(x)= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega}^{\cdot} \nabla . u(y)\, E_{\beta}\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy \ldots \quad (7.4.3)$$

Theorem 2. From definitions (4.1) and (4.3), we have for any u(x): Ω → R, such that

$$\nabla u(x).\, \mathrm{n}|_{\vartheta\Omega} = 0 \ldots \quad (7.4.4)$$

the following identity

$$\nabla \cdot \nabla^{(\alpha)} u(x) = \nabla^{(\alpha)} \cdot \nabla u(x) \qquad \ldots \quad (7.4.5)$$

Proof. Employing (7.4.1), we obtain

$$\nabla . \nabla^{(\alpha)}u(x) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}} \int_{\Omega}^{\cdot} \nabla u(y).\nabla_x\, E_{\beta}\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right] dy$$

$$\nabla.\nabla^{(\alpha)}\mathrm{u}(\mathrm{x}\,) = -\frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}\int_{\Omega}^{\cdot}\nabla u(y).\nabla E_{\beta}\left[-\frac{\alpha^{2}(x-y)^{2}}{(1-\alpha)^{2}}\right]dy\ldots \quad (7.4.6)$$

$$= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}\int_{\Omega}^{\cdot}\nabla.\nabla u(y)\,E_{\beta}\left[-\frac{\alpha^{2}(x-y)^{2}}{(1-\alpha)^{2}}\right]dy -$$
$$\frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}\int_{\partial\Omega}^{\cdot}\nabla u(y).\mathrm{n}\,E_{\beta}\left[-\frac{\alpha^{2}(x-y)^{2}}{(1-\alpha)^{2}}\right]dy$$

hence, from the boundary condition (7.4.4), the identity (7.4.5) is proved, be

cause (7.4.6) coincides with$\nabla^{(\alpha)}.\nabla\mathrm{u}(\mathrm{x}\,)$ =

$$\frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}\int_{\Omega}^{\cdot}\nabla.\nabla u(y)\,E_{\beta}\left[-\frac{\alpha^{2}(x-y)^{2}}{(1-\alpha)^{2}}\right]\,dy$$

7.5 Fourier transform of fractional gradient and divergence

For a smooth function u(x) :R^{3} → R the Fourier transform (FT) of the fracti onal gradient is defined by FT($\nabla^{(\alpha)}u(x)$) (ξ) = $\int_{R^{3}}^{\cdot}\nabla^{(\alpha)}\,u(x)\,E_{\beta}[-2\pi i\xi.x]\,dx$

Thus, if we consider the gradient of (4.1), the Fourier transform is given by

$$FT(\nabla^{(\alpha)}u)(\xi) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}\;FT\left(\int_{R^{3}}^{\cdot}\nabla u(y)\,E_{\beta}\left[-\frac{\alpha^{2}(x-y)^{2}}{(1-\alpha)^{2}}\right]dy\right)(\xi)$$

$$= \frac{\alpha}{(1-\alpha)\sqrt{\pi^{\alpha}}}\;FT(\nabla u)(\xi)FT(E_{\beta}\left[-\frac{\alpha^{2}(x-y)^{2}}{(1-\alpha)^{2}}\right]\;)\;(\xi)$$

where

$$FT(E_\beta\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right]\ )\ \ (\xi)=\frac{(1-\alpha)\sqrt{\pi}}{\alpha}E_\beta\left[\frac{\pi^2(1-\alpha^2)\xi^2}{\alpha^2}\right]$$

Thus, we obtain:

$$FT(\nabla^{(\alpha)}u)(\xi) = \sqrt{\pi^{1-\alpha}}\mathrm{FT}(\nabla \mathrm{u})(\xi)E_\beta\left[-\frac{\pi^2(1-\alpha^2)\xi^2}{\alpha^2}\right]$$

The Fourier transform of fractional divergence is defined by

$$FT(\nabla^{(\alpha)}u)(\xi) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}}\ FT\left(\int_\Omega^{\cdot} \nabla u(y)\, E_\beta\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right]dy\right)(\xi)$$

from which we have

$$FT(\nabla^{(\alpha)}u)(\xi) = \sqrt{\pi^{1-\alpha}}\mathrm{FT}(\nabla \mathrm{u})(\xi)E_\beta\left[-\frac{\pi^2(1-\alpha^2)\xi^2}{\alpha^2}\right]$$

7.6 Fractional Laplacian:

In the study of partial differential equations, there is a great interest in fracti onal Laplacian. Using the definitions of fractional gradient and divergence, we can suggest the representation of fractional Laplacian for a smooth funct ion f(x): $\Omega \rightarrow R^3$, such that ∇ f(x)· $n|_{\vartheta\Omega}$= 0, as

$$(\nabla^2)^{(\alpha)}\mathrm{f}\ (\mathrm{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}}\int_\Omega^{\cdot} \nabla.\nabla\ \mathrm{f}(y)\, E_\beta\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right]dy$$

By Theorem 2.1, we have $(\nabla^2)^{(\alpha)}$ f (x) = ∇· $(\nabla)^{(\alpha)}$ f (x) = $(\nabla)^{(\alpha)}\cdot$ ∇f (x)

Now, we suppose that

f(x) = 0 , on $\partial\Omega$

then we extend the function f(x) = 0 on $R^3\backslash\Omega$. So, we consider the Fourier t ransform

$$\mathrm{FT}\ ((\nabla^2)^{(\alpha)}\mathrm{f}\ (\mathrm{x})) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}}\ FT\left(\int_{R^3}^{\cdot} \nabla^2\ \mathrm{f}(y)\, E_\beta\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right]dy\right)(\xi)\ldots\ (7.6.1)$$

$$=\frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}}\ FT\big(\nabla.\nabla f(x)\big)(\xi)FT\left(E_\beta\left[-\frac{\alpha^2x^2}{(1-\alpha)^2}\right]\right)(\xi)$$

$= 4\pi|\xi|^2 FT\big(f(x)\big)(\xi)\sqrt{\pi^{1-\alpha}}E_\beta\left[-\frac{(1-\alpha)^2\xi^2}{\alpha^2}\right]$

Finally, if $\alpha = 1$ we obtain from (6.1)

$$\text{FT }(\nabla^2 f(x)) = -\lim_{\substack{\alpha\to 1\\ k\to 1}} 4\pi|\xi|^2 FT\big(f(x)\big)(\xi)\sqrt{\pi^{1-\alpha}}\,E_\beta\left[-\frac{(1-\alpha)^2\xi^2}{\alpha^2}\right]$$

$= -\,4\pi|\xi|^2 LT\big(f(x)\big)(\xi).$

Conclusion: